ELSEVIER Journal of Geometry and Physics 28 (1998) 94–128

JOURNAL OF GEOMETRY AND PHYSICS

Quantum and braided diffeomorphism groups

Shahn Majid [1]

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, UK

Received 3 February 1998

Abstract

We develop a general theory of 'quantum' diffeomorphism groups based on the universal comeasuring quantum group $M(A)$ associated to an algebra A, and its various quotients. Explicit formulae are introduced for this construction, as well as dual quasi-triangular and braided R-matrix versions. Among the examples, we construct the q-diffeomorphisms of the quantum plane $yx = qxy$, and recover the quantum matrices $M_q(2)$ as q-diffeomorphisms respecting its braided group addition law. © 1998 Elsevier Science B.V. All rights reserved.

Subj. Class.: Quantum groups
1991 MSC: 17B37; 81R50; 58D05
Keywords: Quantum groups; Braided groups; Diffeomorphism groups

1. Introduction

In [1] one finds a standard construction for a measuring coalgebra $M(A, B)$ as a universal object for coalgebras 'acting' form algebras A to B. The diagonal case $M(A, A)$ is a bialgebra. The construction is the analogue of the classical automorphism group except now as the universal or 'maximal' object in the category of bialgebras rather than of groups. Until now, however, this measuring bialgebra construction has been little studied in the modern quantum groups literature, although see [2,3]. Probably the main reason for this is the lack of explicit formulae: being defined as a universal object it is generally hard to compute.

[1] Royal Society University Research Fellow and Fellow of Pembroke College, Cambridge; UK. Tel.: +44 1 223 337900; fax: +44 1 223 337918; e-mail: majid@damtp.cam.ac.uk.

PII: S0393-0440(98)00016-3

In the present paper we develop a much more explicit and computable version of this construction, namely a bialgebra $M(A)$ defined directly by the structure constants of an associative algebra A, which we call the 'comeasuring bialgebra'. It is the arrows-reversed version of the standard construction but, by contrast, is defined by the generators and relations in a familiar way. It has many applications and, in fact, a quasi-quantum group version is used in our forthcoming paper [4] with H. Albuquerque as a definition of the automorphism object of quasi-associative algebras such as the octonions. By contrast, here we study in much more detail the strictly associative setting and its 'geometrical' applications. We also generalise the construction to the general braided case, where A is an algebra in a braided category such as that associated to a Yang–Baxter matrix. Such algebras abound in the theory of q-deformations as the 'geometrical objects' on which quantum groups act as symmetries.

The automorphism object $M(A)$ clearly plays the role of 'diffeomorphisms' in a noncommutative geometry setting (where an algebra A is viewed as like the 'functions' on some space). This is our point of view, and we will use the corresponding terminology throughout the paper. We also consider briefly quotients of $M(A)$ that preserve a given differential structure on A, but for the most part we work with the universal differential calculus canonically associated to an algebra. Also, we use the terms'quantum group' and 'braided group' a little loosely, without requiring the existence of an antipode or 'group inversion'. In examples, the restricted comeasuring bialgebras $M_0(A)$ will often have an antipode or will admit one by adjoining inverse determinants etc. Within these limitations, we provide a general approach to quantum diffeomorphisms which includes q-diffeomorphisms of the line (i.e. some version of a q-Virasoro quantum group) and of the quantum plane, as well as of finite-dimensional algebras. Our approach is further justified by showing that elements of 'quantum geometry' may be built around these objects, along the lines of [5,6]. The notion of quantum diffeomorphism group should be viewed as a step towards the notion of 'quantum manifold', which is a long-term motivation for the present work.

An outline is the following. In the preliminary Section 2 we formulate the arrows-reversed measuring bialgebra construction and obtain explicit formulae for it and its natural quotients. Apart from the forthcoming paper [4] in the quasi-associative setting, these formulae appear to be new. In Section 3 we compute several examples of comeasuring algebras, exploring their role as 'quantum diffeomorphisms' of polynomial and discrete spaces. Section 3.7 contains the maximal comeasuring bialgebra of the quantum plane, while Section 3.8 obtains the 2×2 quantum matrices $M_q(2)$ as the q-diffeomorphisms respecting addition. Note that this result is somewhat different from the characterisation of quantum matrices in [7], which is based on a certain construction of 'endomorphisms' for quadratic algebras and their duals. There are some points of contact, however. In Section 4 we turn to general R-matrix constructions. We give a dual quasi-triangular version $M(R, A)$ of the comeasuring construction, including the applications to the line and the quantum plane. Also using R-matrices in this section are braided group versions of our constructions.

2. General constructions

The abstract definition of the comeasuring bialgebra is obtained from [1] by reversing arrows, giving a universal comeasuring algebra $M(A, B)$ 'coacting' from algebras A to B. Here the arrows of the coacted-upon objects A, B are not reversed, i.e. we leave these as algebras and do not make them into coalgebras as a full dualisation would do. Also, we concentrate on the diagonal case $M(A) = M(A, A)$ since we will not have much to say about the general non-diagonal case. Their formulation is, however, strictly analogous. We work over a general field k.

Thus, by definition, a *comeasuring* of a unital algebra A is a pair (B, β) where B is a unital algebra and $\beta : A \to A \otimes B$ is an algebra map to the tensor product algebra. We define $(M(A), \beta_U)$, when it exists, to be the initial object in the category of comeasurings of A, i.e. a comeasuring such that for any (B, β) there exists a unique algebra map $\pi : M(A) \to B$ such that $\beta = (\mathrm{id} \otimes \pi)\beta_U$.

Proposition 2.1 (cf. [1]). *$M(A)$, when it exists, is a bialgebra and β_U is a coaction of it on A as an algebra. Any other coaction of a bialgebra on A as an algebra is a quotient of this one.*

Proof. This is elementary. We note that $M(A)\otimes M(A)$, $(\beta_U \otimes \mathrm{id})\circ\beta_U$ is also a comeasuring. Hence there is an algebra map $\Delta : M(A) \to M(A) \otimes M(A)$ and $(\beta_U \otimes \mathrm{id}) \circ \beta_U = (\mathrm{id} \otimes \Delta) \circ \beta_U$. It remains to show that Δ is coassociative. For this, consider $M(A)^{\otimes 3}$, $(\beta_U \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\beta_U \otimes \mathrm{id}) \circ \beta_U$ as another comeasuring. The map $\pi : M(A) \to M(A)^{\otimes 3}$ in this case is such that $(\mathrm{id} \otimes \pi) \circ \beta_U$ is the comeasuring map associated to $M(A)^{\otimes 3}$. Both $(\Delta \otimes \mathrm{id}) \circ \Delta$ and $(\mathrm{id} \otimes \Delta) \circ \Delta$ clearly fulfill the role of π, and since π is unique, these maps coincide. Finally, k, $\beta(a) = a \otimes 1$ is a comeasuring and $\epsilon : M(A) \to k$ is the induced map. It is easy to see that it provides a counit. Given any other coaction of a bialgebra B on A as an algebra (i.e A a B-comodule algebra), the fact that B comeasures gives the required map $\pi : M(A) \to B$. □

When A is nonunital, we can follow the same definitions while omitting the conditions that B is unitial and that β, π respect the unit. In this case it is clear that the universal object $M(A)$ is a not-necessarily unital bialgebra. We call it the 'nonunital version' of the comeasuring bialgebra. We can, however, always formally adjoin a unit to it, extending Δ, ϵ by $\Delta(1) = 1 \otimes 1$ and $\epsilon(1) = 1$. We denote this extension by $M_1(A)$.

Suppose now that A is finite-dimensional and let $\{e_i\}$ be a basis. We let $e_i e_j = c_{ij}{}^k e_k$ define its structure constants.

Proposition 2.2. *$M_1(A)$ is generated by 1 and a matrix $\mathbf{t} = (t^i{}_j)$ of generators, with relations and coproduct*

$$c_{ij}{}^a t^k{}_a = c_{ab}{}^k t^a{}_i t^b{}_j, \quad \Delta t^i{}_j = t^i{}_a \otimes t^a{}_j, \quad \epsilon(t^i{}_j) = \delta^i{}_j.$$

The map $\beta_U(e_i) = e_a \otimes t^a{}_i$ is the coaction.

Proof. It is easy to verify that Δ extends as an algebra map and that β_U is a coaction. For the former, the proof is

$$\Delta(c_{ab}{}^k t^a{}_i t^b{}_j) = c_{ab}{}^k t^a{}_m t^b{}_n \otimes t^m{}_i t^n{}_j = c_{mn}{}^a t^k{}_a \otimes t^m{}_i t^n{}_j$$
$$= c_{ij}{}^b t^k{}_a \otimes t^a{}_b = \Delta(c_{ij}{}^a t^k{}_a).$$

(This is also a special case of the quasi-Hopf algebra construction in [4] or of the braided case in Section 4). Now, let (B, β) be a comeasuring and define $\pi(t^i{}_j) \in B$ by $\beta(e_i) = e_a \otimes \pi(t^a{}_i)$. Then β is an algebra map is the assertion that $c_{ij}{}^a \pi(t^k{}_a) = c_{ab}{}^k \pi(t^a{}_i)\pi(t^b{}_j)$, i.e. $\pi : M(A) \to B$ defined in this way extends as an algebra map. Thus $M(A)$ as generated by the $t^i{}_j$ has the required universal property. When we adjoin 1, we obtain $M_1(A)$ as stated. □

Note that the explicit proof that Δ provides a bialgebra works similarly for *any* tensor $c_{i_1 \cdots i_m}{}^{j_1 \cdots j_n}$ of rank (m, n), i.e. one has an associated bialgebra with the matrix coalgebra and the relations

$$c_{a_1 \cdots a_m}{}^{i_1 \cdots i_n} t^{a_1}{}_{j_1} \cdots t^{a_m}{}_{j_m} = t^{i_1}{}_{a_1} \cdots t^{i_n}{}_{a_n} c_{j_1 \cdots j_m}{}^{a_1 \cdots a_n}. \tag{1}$$

The associated bialgebra may, however, be trivial. In our case the associativity of A, which is the equation $c_{ij}{}^a c_{ak}{}^l = c_{ia}{}^l c_{jk}{}^a$, is needed for the interpretation as universal comeasuring object. It also tends to ensure that the ideal generated by the stated relations is not too big. Specifically,

$$c_{ab}{}^c (t^a{}_i t^b{}_j) t^d{}_k c_{cd}{}^l = c_{ij}{}^b t^c{}_b t^a{}_k c_{ca}{}^l = c_{ij}{}^b c_{bk}{}^a t^l{}_a = c_{ib}{}^a t^l{}_a c_{jk}{}^b$$
$$= c_{ac}{}^l t^a{}_i t^c{}_b c_{jk}{}^b = c_{ac}{}^l c_{bd}{}^c t^a{}_i (t^b{}_j t^d{}_k)$$
$$= c_{ab}{}^c t^a{}_i (t^b{}_j t^d{}_k) c_{cd}{}^l$$

holds automatically in $M_1(A)$.

Now suppose that A is unital and $e_0 = 1$ is a basis element. We let $e_i, i = 1, \ldots, \dim(A) - 1$ be the remaining basis elements.

Proposition 2.3. *The comeasuring bialgebra $M(A)$ is the quotient of $M_1(A)$ by $t^0{}_0 = 1$, $t^i{}_0 = 0$. Explicitly, it has the relations*

$$c_{ij}{}^a t^k{}_a = c_{ab}{}^k t^a{}_i t^b{}_j + b_i t^k{}_j + t^k{}_i b_j,$$
$$c_{ij}{}^0 + c_{ij}{}^a b_a = c_{ab}{}^0 t^a{}_i t^b{}_j + b_i b_j$$

and coproduct, coaction

$$\Delta t^i{}_j = t^i{}_a \otimes t^a{}_j, \qquad \Delta b_i = b_a \otimes t^a{}_i + 1 \otimes b_i,$$
$$\epsilon(t^i{}_j) = \delta^i{}_j, \qquad \epsilon(b_i) = 0$$
$$\beta_U(e_i) = 1 \otimes b_i + e_a \otimes t^a{}_i,$$

where $b_i = t^0{}_i$.

Proof. Here $t^0{}_0 - 1$, $t^i{}_0$ generates a biideal of $M_1(A)$ and hence provides a quotient bialgebra. For a direct proof that Δ as stated extends to products as an algebra map, we have

$$\begin{aligned}\Delta c_{ij}{}^a t^k{}_a &= c_{ij}{}^a t^k{}_b \otimes t^b{}_a\\ &= c_{ab}{}^c t^k{}_c \otimes t^a{}_i t^b{}_j + t^k{}_c \otimes b_i t^c{}_j + t^k{}_c \otimes t^c{}_i b_j\\ &= c_{cd}{}^k t^c{}_a t^d{}_b \otimes t^a{}_i t^b{}_j + b_a t^k{}_b \otimes t^a{}_i t^b{}_j\\ &\quad + t^k{}_a b_b \otimes t^a{}_i t^b{}_j + t^k{}_c \otimes b_i t^c{}_j + t^k c \otimes t^c{}_i b_j\\ &= \Delta(c_{ab}{}^k t^a{}_i t^b{}_j + b_i t^k{}_j + t^k{}_i b_j)\end{aligned}$$

using the relations and coproduct stated for $M(A)$. Similarly,

$$\begin{aligned}\Delta(c_{ab}{}^0 t^a{}_i t^b{}_j + b_i b_j) &= c_{ab}{}^0 t^a{}_c t^b{}_d \otimes t^c{}_i t^d{}_j + b_a b_b \otimes t^a{}_i t^b{}_j\\ &\quad + 1 \otimes b_i b_j + b_a \otimes t^a{}_i b_j + b_a \otimes b_i t^a{}_j\\ &= 1 \otimes c_{ab}{}^0 t^a{}_i t^b{}_j + 1 \otimes b_i b_j + b_a \otimes c_{ij}{}^b t^a{}_b\\ &= c_{ij}{}^0 1 \otimes 1 + c_{ij}{}^a 1 \otimes b_a + c_{ij}{}^b b_a \otimes t^a{}_b\\ &= \Delta(c_{ij}{}^0 + c_{ij}{}^a b_a)\end{aligned}$$

for the second set of relations. The coproduct has a standard form which is clearly coassociative, hence we obtain a bialgebra. It inherits the coaction as shown. Conversely, suppose that B, β is a comeasuring and let $\pi(b_i), \pi(t^i{}_j) \in B$ defined by $\beta(e_i) = 1 \otimes \pi(b_i) + e_a \otimes \pi(t^a{}_i)$. Similarly to the nonunital case, it follows from β a unital algebra map that π extends as a unital algebra map and $\beta = (\mathrm{id} \otimes \pi) \circ \beta_U$ by construction. □

Note that these constructions are independent of the choice of basis (beyond the choice $e_0 = 1$ in the unital case) because they are abstractly defined. In a new basis $e'_i = e_a \Lambda^a{}_i$, the generators of $M_1(A)$ are

$$t'^i{}_j = \Lambda^{-1i}{}_a t^a{}_b \Lambda^b{}_j. \tag{2}$$

In the unital case we require that the transformation is of the form $e'_0 = e_0$ and $e'_i = e_0 \lambda_i + e_a \Lambda^a{}_i$ where the indices on λ, Λ run from $1, \ldots, \dim(A) - 1$. Then the transformed generators of $M(A)$ are

$$b'_i = b_a \Lambda^a{}_i + \lambda_i - \lambda_b \Lambda^{-1b}{}_c t^c{}_d \Lambda^d{}_i, \qquad t'^i{}_j = \Lambda^{-1i}{}_a t^a{}_b \Lambda^b{}_j. \tag{3}$$

If we are given slightly more structure, namely a linear splitting $A = 1 \oplus A'$ then we can define a restricted comeasuring bialgebra $M_0(A)$ as the universal object for comeasurings that respect the splitting, i.e. such that $\beta(A') \subset A' \otimes B$ in addition to $\beta(1) = 1$ as before.

Proposition 2.4. *$M_0(A)$ is the quotient of $M(A)$ by $b_i = 0$, i.e. it is the associative algebra generated by 1, $t^i{}_j$ where $i, j = 1, \ldots, \dim(A) - 1$, and the relations*

$$c_{ij}{}^a t^k{}_a = c_{ab}{}^k t^a{}_i t^b{}_j, \qquad c_{ij}{}^0 = c_{ab}{}^0 t^a{}_i t^b{}_j$$

and matrix coalgebra. Its coaction is $\beta_U(e_i) = e_a \otimes t^a{}_i$.

Proof. We suppose that a splitting $A = 1 \oplus A'$ is given and $\{e_i\}$ are a basis of A'. We clearly have a biideal of $M(A)$ generated by the b_i and hence a quotient bialgebra. It inherits the relations and coaction shown. Conversely, if (B, β) is a comeasuring preserving A', we define $\pi(t^i{}_j) \in B$ by $\beta(e_i) = e_a \otimes \pi(t^a{}_i)$ and $\pi(1) = 1$. □

This depends on the basis only through the splitting of 1 defined by it, i.e. is independent of the basis of A' (one may take transformations as above with $\lambda = 0$). In summary, associated to a split unital algebra A, we have a sequence of bialgebras

$$M_1(A) \to M(A) \to M_0(A),$$

where $M(A)$ is the 'usual' definition dual to the construction in [1], respecting the unit but not its splitting, $M_1(A)$ is the unital extension of the nonunital version, and $M_0(A)$ is the version respecting the unit and in addition its splitting.

We have similar formulae for $M_1(A, B)$, $M(A, B)$, $M_0(A, B)$ as algebras of 'maps' between algebras. For example, if B has structure constants $d_{\alpha\beta}{}^{\gamma}$ in basis $\{f_\alpha\}$, then $M_1(A, B)$ is generated by 1 and a rectangular matrix $\{t^i{}_\alpha\}$ of generators with relations

$$c_{ij}{}^k t^i{}_\alpha t^j{}_\beta = d_{\alpha\beta}{}^{\gamma} t^k{}_\gamma.$$

Similarly for its quotients. Also, although we have assumed finite bases for our explicit formulae, similar formulae hold more generally with countable basis. In this case one may have to work formally or, more precisely, with a suitable completion of tensor products. In this respect, the original measuring (rather than comeasuring) construction in [1] is better behaved. On the other hand, we have the advantage in the dual formulation of explicit formulae as algebras with generators and relations.

3. Quantum diffeomorphisms of polynomial algebras

We now compute the comeasuring bialgebras for polynomials $\mathbb{C}[x]$ and their quotients, and justify their role as 'diffeomorphisms'. Further justification is in Section 3.6 where we consider the preservation of nonuniversal differential calculi; for the moment the underlying calculus is the universal one canonically associated to the algebra. In the case of $\mathbb{C}[x]$, the full comeasuring bialgebras $M_1(\mathbb{C}[x])$ and $M(\mathbb{C}[x])$ require formal powerseries and are included for motivation only, but $M_0(\mathbb{C}[x])$ is a completely algebraic object.

Clearly, the algebraic version of a diffeomorphisms $\mathbb{C}[x] \to \mathbb{C}[x]$ is a polynomial map $x \mapsto a(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, i.e. a polynomial. If we compose two such diffeomorphisms a, b, then

$$\begin{aligned} a \circ b(x) &= \sum_j a(x^j) b_j = \sum_j \sum_{n_1,\ldots,n_j} x^{n_1+\cdots+n_j} a_{n_1} \cdots a_{n_j} b_j \\ &= \sum_i x^i \sum_j \sum_{n_1+\cdots+n_j=i} a_{n_1} \cdots a_{n_j} b_j, \end{aligned}$$

where $i, j, n_1 \in \mathbb{Z}_+$, etc. Here $\mathbb{Z}_+$ denotes the natural numbers including 0. On the other hand, there is not necessarily a polynomial inverse. Therefore the 'coordinate ring' of the semigroup of such diffeomorphisms is the commutative bialgebra $\mathrm{Diff}(\mathbb{C}[x]) = \mathbb{C}[t_i | i \in \mathbb{Z}_+]$ with countable generators t_i and the coaction

$$x \mapsto \sum_i x^i \otimes t_i \tag{4}$$

as a formal power series. The corresponding coproduct and counit is

$$\Delta t_i = \sum_j \sum_{n_1+\cdots+n_j=i} t_{n_1} \cdots t_{n_j} \otimes t_j, \qquad \epsilon(t_i) = \delta_{i,1} \tag{5}$$

and also involves powerseries. On the other hand, if we restrict attention to diffeomorphisms which preserve the point zero in the line, we consider only polynomials $a(x) = a_1 x + a_2 x^2 + \cdots$, i.e. without constant term. This restricted diffeomorphism bialgebra is the commutative bialgebra $\mathrm{Diff}_0(\mathbb{C}[x]) = \mathbb{C}[t_i | i \in \mathbb{N}]$ with the same formulae as above but with $i, j, n_1 \in \mathbb{N}$ etc. Here $\mathbb{N}$ denotes the natural numbers not including 0. In this case (5) has only a finite number of terms. Thus the algebraic diffeomorphism fixing 0 form a bialgebra with algebraic coalgebra structure and countable generators.

3.1. Comeasurings of the line

We now apply the constructions in Section 2, and recover (noncommutative) versions of the above diffeomorphism bialgebras as $M(\mathbb{C}[x])$ and $M_0(\mathbb{C}[x])$, respectively. Thus, we take $A = \mathbb{C}[x]$ and basis $e_0 = 1$ and $e_i = x^i$ for $i \in \mathbb{N}$. Then the structure constants are

$$c_{ij}{}^k = \delta^k_{i+j}, \qquad c_{ij}{}^0 = 0 \tag{6}$$

and the comeasuring bialgebra $M(A)$ from Proposition 2.3 has generators $1, b_i$ and $t^i{}_j$ with relations

$$b_{i+j} = b_i b_j, \qquad t^k{}_{i+j} = \sum_{m+n=k} t^m{}_i t^n{}_j + b_i t^k{}_j + t^k{}_i b_j \tag{7}$$

for all indices in $\mathbb{N}$. These relations allow is to consider $t_0 \equiv b_1$ and $t_i \equiv t^i{}_1$ as the generators, obtaining the other $b_i, t^i{}_j$ inductively as

$$t^i{}_j = \sum_{n_1+\cdots+n_j=i} t_{n_1} \cdots t_{n_j}, \quad b_i = (t_0)^i \tag{8}$$

for $i, j \in \mathbb{N}$ and $n_1 \cdots n_j \in \mathbb{Z}_+$. In this case relations (7) become empty, so $M(\mathbb{C}[x]) = \mathbb{C}\langle t_i | i \in \mathbb{Z}_+\rangle$, the free algebra on countable generators. Its coproduct and coaction from Proposition 2.3 therefore takes the form (5) and (4) with all indices from $\mathbb{Z}_+$, i.e. $M(\mathbb{C}[x])$ has the same form as Diff $(\mathbb{C}[x])$ except that the generators t_i are totally noncommuting.

The restricted comeasuring bialgebra $M_0(\mathbb{C}[x])$ is the quotient of this where we set t_0=0. Equivalently, working from Proposition 2.4, it has generators $t^i{}_j$ for $i, j \in \mathbb{N}$ with relations

$$\sum_{m+n=k} t^m{}_i t^n{}_j = t^k{}_{i+j}, \tag{9}$$

where all indices are in $\mathbb{N}$. As before, this implies

$$t^i{}_j = \sum_{n_1+\cdots+n_j=i} t_{n_1}\cdots t_{n_j}, \tag{10}$$

where, $t_i \equiv t^i{}_1$ and now the $n_1,\ldots,n_j \in \mathbb{N}$. This implies in particular that $t^i{}_j = 0$ for $j > i$, which ensures that the matrix coproduct is now a finite sum of terms. In this way, $M_0(\mathbb{C}[x]) = \mathbb{C}\langle t_i | i \in \mathbb{N}\rangle$, the free algebra, with coproduct (5) and coaction (4) where all indices are in $\mathbb{N}$. Thus, $M_0(\mathbb{C}[x])$ has the same form as $\mathrm{Diff}_0(\mathbb{C}[x])$ except that its generators are totally noncommuting.

Finally, the extended comeasuring bialgebra $M_1(\mathbb{C}[x])$ has no classical analogue (since usual diffeomorphisms preserve the constant function $1 \in \mathbb{C}[x]$). From Proposition 2.2 it is generated by 1 and $t^i{}_j$, where $i, j \in \mathbb{Z}_+$ and relations (9) (but now with all indices in $\mathbb{Z}_+$). This time we can reduce the matrix generators to two sequences $E_i \equiv t^i{}_0$ and $t_i \equiv t^i{}_1$ for $i \in \mathbb{Z}_+$. The others are recovered by (8) where now $i, n_1 \in \mathbb{Z}_+$ etc., and $j \in \mathbb{N}$. In this way we find that $M_1(\mathbb{C}[x])$ is generated by 1, E_i, t_i for $i \in \mathbb{Z}_+$, with the residual relations and coalgebra

$$\sum_{m+n=i} E_m E_n = E_i, \qquad \sum_{m+n=i} E_m t_n = t_i = \sum_{m+n=i} t_m E_n,$$
$$\Delta E_i = \sum_j \sum_{n_1+\cdots+n_j=i} t_{n_1}\cdots t_{n_j} \otimes E_j,$$
$$\Delta t_i = \sum_j \sum_{n_1+\cdots+n_j=i} t_{n_1}\cdots t_{n_j} \otimes t_j$$

and $\epsilon(E_i) = \delta^i{}_0$, $\epsilon(t_i) = \delta^i{}_1$, where all indices are from $\mathbb{Z}_+$. We note that the constrained sums in these expressions are the usual convolution product $*$ on sequences, and in that notation the relations and coproduct of $M_1(\mathbb{C}[x])$ are

$$E * E = E, \qquad E * t = t = t * E,$$
$$\Delta E_i = \sum_j (t * \cdots * t)_i \otimes E_j, \qquad \Delta t_i = \sum_j (t * \cdots * t)_i \otimes t_j,$$

where the $*$-products are j-fold. Finally, the coaction is

$$1 \mapsto \sum_i x^i \otimes E_i, \qquad x \mapsto \sum_i x^i \otimes t_i.$$

The $M(\mathbb{C}[x])$ above is the quotient of this non-unit-preserving diffeomorphism group by setting $E_i = \delta^i{}_0$.

3.2. Comeasurings of Grassmann and anyonic variables

Here we consider 'diffeomorphisms' of the finite-dimensional quotients $A = \mathbb{C}[x]/x^N = 0$. The case $N = 2$ is where x is a fermionic or 'Grassmann' variable. The case $N > 2$ is an anyonic variable. The addition law and geometry of this 'anyonic line' can be found in [8].

We take basis $e_i = x^i$ where $i = 1, \ldots, N-1$ and $e_0 = 1$. Then $c_{ij}{}^k$ have the same form (6) but restricted in range to $0 < i, j, k < N$. In this case $M = \mathbb{C}\langle t_i \mid i = 0, \ldots, N-1\rangle$ is the free algebra and has coproduct (5) with indices including 0. The restricted comeasuring bialgebra is the free algebra $M_0 = \mathbb{C}\langle t_i \mid i = 1, \ldots, N-1\rangle$ with coproduct likewise from (5) but with all indices now excluding 0. On the other hand the comeasuring bialgebra is now finitely generated and does not need any formal powerseries. (One can then obtain $M(\mathbb{C}[x])$ in the projective limit $N \to \infty$.)

For the fermionic case $x^2 = 0$, the comeasuring bialgebra is $M = \mathbb{C}\langle b, t\rangle$ with

$$\Delta b = 1 \otimes b + b \otimes t, \quad \Delta t = t \otimes t, \quad \epsilon(b) = 0, \quad \epsilon(t) = 1,$$

i.e. the matrix of generators has the form

$$\begin{pmatrix} 1 & b \\ 0 & t \end{pmatrix}.$$

The restricted comeasuring bialgebra is just $M_0 = \mathbb{C}[t]$ with $\Delta t = t \otimes t$ and $\epsilon(t) = 1$. The extended comeasuring bialgebra M_1 is generated by 1 and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with the relations

$$a^2 = a, \quad ac + ca = c, \qquad ab = b = ba, \qquad ad + cb = d = da + bc$$

and the usual matrix coalgebra.

For an anyonic variable with $x^3 = 0$, the comeasuring bialgebra is $M = \mathbb{C}\langle b, t, s\rangle$ with

$$\begin{aligned} &\Delta b = 1 \otimes b + b \otimes t + b^2 \otimes s, && \Delta t = t \otimes t + (bt + tb) \otimes s, \\ &\Delta s = s \otimes t + (t^2 + sb + bs) \otimes s, && \epsilon(b) = \epsilon(s) = 0, \quad \epsilon(t) = 1, \end{aligned} \tag{11}$$

i.e. the matrix of generators takes the form

$$\begin{pmatrix} 1 & b & b^2 \\ 0 & t & bt + tb \\ 0 & s & t^2 + sb + bs \end{pmatrix}.$$

The restricted comeasuring bialgebra is $M_0 = \mathbb{C}\langle t, s\rangle$ with

$$\Delta t = t \otimes t, \quad \Delta s = s \otimes t + t^2 \otimes s, \quad \epsilon(t) = 1, \quad \epsilon(s) = 0, \tag{12}$$

i.e. the matrix of generators take the form

$$\begin{pmatrix} t & 0 \\ s & t^2 \end{pmatrix}.$$

3.3. Comeasurings of roots of unity

For another class of discrete spaces (which we consider as discrete models of a circle), we consider the algebra $A = \mathbb{C}\mathbb{Z}_N = \mathbb{C}[x]/x^N = 1$. We take basis $e_0 = 1$ and $e_i = x^i$ as before, with

$$c_{ij}{}^k = \delta^k{}_{i+j}, \qquad c_{ij}{}^0 = \delta_{i,-j},$$

where $0 < i, j, k < N$ are added modulo N. One may put these into Propositions 2.2–2.4 to obtain the comeasuring bialgebras.

Actually, it is easier in this class of examples to compute M_1 first and then quotient it. This has generators $1, t^i{}_j$ with relations (9), where $i, j \in \mathbb{Z}_N$ including zero. The convolution formula (8) now applies for all (i, j) since $\mathbb{Z}_N$ is a group, so generators may all be written in terms of $t_i \equiv t^i{}_1$ for $i \in \mathbb{Z}_n$. In particular,

$$E_i \equiv t^i{}_0 = \sum_{n_1+\cdots+n_N=i} t_{n_1}\cdots t_{n_N},$$

where all indices are in $\mathbb{Z}_N$. Thus, the extended comeasuring bialgebra M_1 is generated by $1, t_i$ for $i \in \mathbb{Z}_N$ with the relations

$$\sum_{n_1+\cdots+n_{N+1}=i} t_{n_1}\cdots t_{n_{N+1}} = t_i \quad \forall i \in \mathbb{Z}_N$$

and coproduct (5) with all indices in $\mathbb{Z}_N$.

The comeasuring algebra M is given by setting $E_i = \delta^i{}_0$. Hence it is generated by $1, t_i$ for $i \in \mathbb{Z}_N$ modulo the relations

$$\sum_{n_1+\cdots+n_N=i} t_{n_1}\cdots t_{n_N} = \delta^i{}_0 \qquad \forall i \in \mathbb{Z}_N$$

and the same form of coproduct. Here

$$b_i \equiv t^0{}_i = \sum_{n_1+\cdots+n_i=0} t_{n_1}\cdots t_{n_i}, \quad t^i{}_j = \sum_{n_1+\cdots+n_j=i} t_{n_1}\cdots t_{n_j}$$

for $i, j = 1, \ldots, N-1$ and $n_1 \in \mathbb{Z}_N$, etc., provide the generators in the form of Proposition 2.3.

Finally, the restricted comeasuring bialgebra M_0 is given by setting $b_i = 0$. Hence M_0 is generated by $1, t_i$ where $i = 1, \ldots, N-1$ modulo the relations

$$\sum_{n_1+\cdots+n_N=0} t_{n_1}\cdots t_{n_N} = 1, \qquad \sum_{n_1+\cdots+n_N=i} t_{n_1}\cdots t_{n_N} = 0,$$

$$\sum_{n_1+\cdots+n_i=0} t_{n_1}\cdots t_{n_i} = 0$$

for i, n_1, etc., are in the range $1, \ldots, N-1$ and addition modulo $\mathbb{Z}_N$. The coproduct is (5) with all indices similarly in this range.

For $x^2 = 1$, the extended comeasuring bialgebra M_1 is generated by $1, b, t$ modulo the relations and coalgebra

$$\begin{aligned} &(b \pm t)^3 = (b \pm t), \qquad \Delta t = t \otimes t + (bt + tb) \otimes b, \\ &\Delta b = (b^2 + t^2) \otimes b + b \otimes t, \qquad \epsilon(t) = 1, \qquad \epsilon(b) = 0. \end{aligned} \tag{13}$$

The matrix of generators has the form

$$\begin{pmatrix} b^2 + t^2 & b \\ bt + tb & t \end{pmatrix}.$$

The comeasuring bialgebra M is obtained by setting $b^2 + t^2 = 1$, $bt + tb = 0$ and is therefore generated by $1, b, t$ modulo the stronger relations and coalgebra

$$\begin{aligned} &(b \pm t)^2 = 1, \qquad \Delta t = t \otimes t, \\ &\Delta b = 1 \otimes b + b \otimes t, \qquad \epsilon(t) = 1, \qquad \epsilon(b) = 0. \end{aligned} \tag{14}$$

Although not a Hopf algebra, there is a formal antipode

$$St = \frac{t}{1 - b^2}, \qquad Sb = -\frac{bt}{1 - b^2}$$

as a powerseries in b. The restricted comeasuring bialgebra is just $M_0 = \mathbb{C}[t]/t^2 = 1$ and $\Delta t = t \otimes t$, $\epsilon(t) = 1$, and is a Hopf algebra with $St = t$.

For $x^3 = 1$, the comeasuring bialgebra M is generated by $1, b, t, s$ modulo the relations

$$b\alpha + t\gamma + s\beta = 1, \quad t\alpha + s\gamma + b\beta = 0, \quad s\alpha + b\gamma + t\beta = 0,$$

where

$$\alpha \equiv b^2 + ts + st, \quad \beta \equiv s^2 + bt + tb, \quad \gamma \equiv t^2 + sb + bs.$$

The coalgebra is

$$\begin{aligned} &\Delta b = 1 \otimes b + b \otimes t + \alpha \otimes s, \qquad \Delta t = t \otimes t + \beta \otimes s, \\ &\Delta s = s \otimes t + \gamma \otimes s, \qquad \epsilon(b) = \epsilon(s) = 0, \qquad \epsilon(t) = 1, \end{aligned}$$

i.e. the matrix of generators has the form

$$\begin{pmatrix} 1 & b & \alpha \\ 0 & t & \beta \\ 0 & s & \gamma \end{pmatrix}.$$

The extended M_1 is similar with weaker relations. Finally, the restricted M_0 is generated by $1, t, s$ with the relations and coproduct

$$\begin{aligned} &ts + st = 0, \quad t^3 + s^3 = 1, \quad t^2 s = ts^2 = 0, \\ &\Delta t = t \otimes t + s^2 \otimes s, \qquad \Delta s = s \otimes t + t^2 \otimes s \end{aligned} \tag{15}$$

and $\epsilon(t) = 1$, $\epsilon(s) = 0$, i.e. the matrix of generators has the form

$$\begin{pmatrix} t & s^2 \\ s & t^2 \end{pmatrix}.$$

The second relation is equivalent to $(t + s)^3 = 1$ given the others.

Similarly, one may compute the measuring bialgebra for $k_\lambda = k[\lambda]/m(\lambda) = 0$ for general fields and general polynomials m. When m is monic and irreducible, k_λ is a field extension of k and $M(k_\lambda)$ should be viewed as the 'quantum Galois group' for the field extension. Also, we are not limited to commutative algebras and fields. The comeasuring bialgebras for the complex numbers and the quaternions, as algebras of dimension 2, 4, respectively, over $\mathbb{R}$ are computed in [4] as preludes to the octonion case. In the quaternion case one has a noncommutative and noncommutative bialgebra projecting into the coordinate ring of SO_3. There are of course plenty of other interesting examples according to one's favourite algebra.

3.4. Comeasurings of finite sets

For completeness, we conclude our collection of general classes of examples with the case $A = \mathbb{C}(\Sigma)$ where Σ is a finite set. In this case we compute only $M_1(\mathbb{C}(\Sigma))$ and $M(\mathbb{C}(\Sigma))$ since there is no particularly natural splitting of the identity without more structure. We take basis $\{e_i = \delta_i, i \in \Sigma\}$, the delta-functions at elements in the set. The structure constants are

$$c_{ij}{}^k = \delta_{i,j}\delta_j^k \tag{16}$$

and hence we find from Section 2 that $M_1(\mathbb{C}(\Sigma))$ is generated by 1 and $\tau^i{}_j$ (say) with relations

$$\tau^k{}_i\tau^k{}_j = \delta_{i,j}\tau^k{}_i \tag{17}$$

(no sum) and the matrix coalgebra structure. This means for each row k the matrix of generators forms an orthogonal family of projectors, i.e. a copy of $\mathbb{C}(\Sigma)$, while there are no relations between the different rows. During the final writing of this paper we learned that such algebras have also been considered in [3] in an interesting C^*-algebra setting, again as some kind of universal automorphism objects for finite sets. In our case we obtain them as an elementary example of the general (but algebraic) construction in Proposition 2.2 dual to [1]. We also have a unit $1 = \sum_i \delta_i$ and the corresponding $M(\mathbb{C}(\Sigma))$ can be computed as follows. We take a different basis $e_0 = 1$ and $\{e_i \mid i \in \Sigma - *\}$ where * is a basepoint (a fixed element of Σ). Then $M(\mathbb{C}(\Sigma))$ is generated by $t^i{}_j, b_i$ with $i, j \in \Sigma - *$ and relations

$$t^k{}_it^k{}_j + b_it^k{}_j + t^k{}_ib_j = \delta_{ij}t^k{}_j, \qquad b_ib_j = \delta_{ij}b_j \tag{18}$$

and the coalgebra in Proposition 2.3. We see that $M(\mathbb{C}(\Sigma)) \supset \mathbb{C}(\Sigma)$ embedded as $\{1, b_i\}$. Note that if we want to describe $M(\mathbb{C}(\Sigma))$ in our original basepoint-free delta-function basis, it consists of the quotient of (17) by the relations

$$\sum_j \tau^i{}_j = 1, \tag{19}$$

where $i, j \in \Sigma$. This is equivalent to (18) via

$$\tau^i{}_j = t^i{}_j + b_j, \qquad \tau^0{}_0 = 1 - \sum_i b_i,$$
$$\tau^0{}_i = b_i, \qquad \tau^i{}_0 = 1 - \sum_j t^i{}_j - \sum_j b_j \tag{20}$$

for $i, j \in \Sigma - *$, which is the transformation induced by the change between the two bases.

We note that the algebras $A = \mathbb{C}[x]/x^N = 1$ are isomorphic by Fourier transform to $\mathbb{C}(\mathbb{Z}_N)$, so their comeasuring bialgebras are isomorphic to those for a finite set with N elements. Explicitly,

$$\tau^i{}_j = N^{-1} \sum_{m,n=0}^{N-1} \mathrm{e}^{(2\pi i/N)(mi-nj)} t^m{}_n \tag{21}$$

is the matrix of projectors (17) generating $M_1(\mathbb{C}(\Sigma))$ in terms of the matrix generators of M_1 for $\mathbb{C}[x]/x^N = 1$ in Section 3.3. For example, for $N = 2$ the matrix of projectors is

$$\tau = \frac{1}{2}\begin{pmatrix} (b+t)^2 + (b+t) & (b+t)^2 - (b+t) \\ (b-t)^2 + (b-t) & (b-t)^2 - (b-t) \end{pmatrix}. \tag{22}$$

That the elements of each row are orthogonal projectors is equivalent to the relations $(b \pm t)^3 = (b \pm t)$ in (13). For the quotient $M(\mathbb{C}(\Sigma))$ we obtain

$$\tau = \begin{pmatrix} g_+ & 1 - g_+ \\ 1 - g_- & g_- \end{pmatrix}, \qquad g_\pm = \frac{1}{2}(1 \pm (b \pm t)) \tag{23}$$

i.e. the algebra generated by two projectors $g_\pm$ and the coalgebra

$$\Delta g_\pm = g_\pm \otimes g_\pm + (1 - g_\pm) \otimes (1 - g_\mp), \qquad \epsilon(g_\pm) = 1 \tag{24}$$

as isomorphic to M for $\mathbb{C}[x]/x^2 = 1$ in the preceding section. This is the basepoint-free description via (17) and (19). On the other hand, if we compute $M(\mathbb{C}(\Sigma))$ from Proposition 2.3 according to (18) (which is in a different basis) we have an equivalent matrix of generators

$$\begin{pmatrix} 1 & {}_\Sigma b \\ 0 & {}_\Sigma t \end{pmatrix}, \quad {}_\Sigma b = 1 - g_+ = \frac{1}{2}(1 - (b+t)), \quad \Sigma^t = g_- - 1 + g_+ = t$$

obeying

$${}_\Sigma b^2 = {}_\Sigma b, \qquad ({}_\Sigma t + {}_\Sigma b)^2 = {}_\Sigma t + {}_\Sigma b,$$
$$\Delta {}_\Sigma b = 1 \otimes {}_\Sigma b + {}_\Sigma b \otimes {}_\Sigma t, \qquad \Delta {}_\Sigma t = {}_\Sigma t \otimes {}_\Sigma t$$

and $\epsilon({}_\Sigma t) = 1, \epsilon({}_\Sigma b) = 0$.

3.5. Elements of quantum geometry

In the remaining two sections we consider briefly some steps towards a 'differential geometry' based on these 'quantum diffeomorphisms'. We essentially use the quantum group approach to noncommutative geometry in [5] and especially the recent paper [6]. We recall that in the classical situation, of Σ is a manifold one has, roughly speaking, an identification $\mathrm{Diff}(\Sigma)/\mathrm{Diff}_*(\Sigma) \cong \Sigma$ providing a principal bundle over Σ with structure group $\mathrm{Diff}_*(\Sigma)$. The canonical projection sends a diffeomorphism σ to $\sigma(*) \in \Sigma$. Moreover, this principal bundle has, again formally, a canonical form θ making this a frame resolution of Σ in the language of [6]. The usual affine frame bundle and linear frame bundle are subbundles. Here θ is the projection to $T_*\Sigma = \mathbb{R}^n$ of the Maurer–Cartan form on $\mathrm{Diff}(\Sigma)$, where the Lie algebra of vector fields on Σ modulo those that vanish at * is identified with the value at $*$. Using the canonical form θ, we have essentially a correspondence between gauge-fields on this principal bundle and covariant derivatives on the cotangent bundle of Σ. In this way, differential geometry on the manifold may be developed strictly as $\mathrm{Diff}_*(\Sigma)$ gauge theory. Of course, because these are large infinite-dimensional groups, one needs topological considerations to make these ideas fully precise in the manifold setting. On the other hand, in our algebraic setting we can try to keep everything algebraic, e.g. if the role of our manifold is played by a finite-dimensional (possibly noncommutative) algebra. We have seen above that for polynomials, $M(\mathbb{C}[x])$ plays the role of 'diffeomorphisms' and $M_0(\mathbb{C}[x])$ plays the role of diffeomorphisms fixing the origin. Motivated by these ideas we make the parallel choice for our finite-dimensional algebras.

To recall the set-up, note that arrows are reversed in our co-ordinate ring formulation. Thus, a homogeneous quantum principal bundle [5] arises from a Hopf algebra map $\pi : M \to M_0$ between Hopf algebras M, M_0 (say) such that the induced coaction $\Delta_L = (\pi \otimes \mathrm{id})\Delta : M \to M_0 \otimes M$ has fixed point subalgebra $A = M^{M_0} \equiv \{h \in M \mid \Delta_L h = 1 \otimes h\}$ (it plays the role of the coordinates of the base manifold), and obeying a certain nondegeneracy condition (the 'Hopf-Galois' condition) that the map

$$\chi : M \otimes_A M \to M_0 \otimes M, \quad h \otimes g \to \Delta_L(h)g$$

is invertible. There is a canonical form $\theta : V \to \Omega^1 M$ given by [6]

$$V = \ker \epsilon|_A, \qquad \theta = (\mathrm{id} \otimes S)\Delta$$

in terms of the Hopf algebra structure of M. Here $\Omega^1 M \subset M \otimes M$ is the kernel of the product map (the universal differential calculus). A connection on the quantum bundle is a map $\omega : M_0 \to \Omega^1 M$ which is left Ad-covariant and obeys $\chi \circ \omega = \mathrm{id} \otimes 1$ on $\ker \epsilon \subset M_0$ and $\omega(1) = 0$ as in [5], and in the presence of θ it defines a covariant derivative $\nabla : \Omega^1 A \to \Omega^1 A \otimes_A \Omega^1 A \cong \Omega^2 A \subset A \otimes A \otimes A$ by

$$\nabla \mathrm{d}a = \mathrm{d}a \otimes 1 - a_{(1)} \otimes Sa_{(2)}\omega(a_{(3)})a_{(4)} \\ - a_{(1)} \otimes Sa_{(2)} \otimes a_{(3)} + 1 \otimes 1 \otimes a,$$

where $\mathrm{d} : A \to \Omega^1 A$ given by $\mathrm{d}a = a \otimes 1 - 1 \otimes a$ is the exterior derivative and $\Delta a = a_{(1)} \otimes a_{(2)}$ etc., is a notation for the coproduct, see [6].

As a first step towards applying this formalism, we consider the case where Σ is a two-point set. We take $M(\mathbb{C}(\Sigma))$ in its basepoint form (18), namely generated by projectors $p \equiv_\Sigma b +_\Sigma t$ and $q \equiv_\Sigma b$ with no further relations and the coalgebra

$$\Delta p = p \otimes p + (1 - p) \otimes q, \qquad \Delta q = (1 - q) \otimes q + q \otimes p,$$
$$\epsilon(p) = 1, \qquad \epsilon(q) = 1.$$

Similarly, $M_0(\mathbb{C}(\Sigma)) = \mathbb{C}[\bar{p}]/\bar{p}^2 = \bar{p}$ with $\Delta\bar{p} = \bar{p}\otimes\bar{p}$ and $\epsilon\bar{p} = 1$. The projection is $\pi(p) = \bar{p}$ and $\pi(q) = 0$. The induced coaction is

$$\Delta_L q = 1\otimes q, \qquad \Delta_L p = \bar{p}\otimes p + (1-\bar{p})\otimes q.$$

Since M is spanned by 1 and alternating words $h = pqpq\cdots$ or $h = qpqp\cdots$, one can show that

$$\Delta_L 1 = 1\otimes 1, \qquad \Delta_L h = \bar{p}\otimes h + (1-\bar{p})\otimes q$$

for either kind of nontrivial h. Hence the fixed point subalgebra is the subalgebra spanned by $1, q$, i.e.

$$M(\mathbb{C}(\Sigma))^{M_0(\mathbb{C}(\Sigma))} = \mathbb{C}(\Sigma).$$

One can also reach a similar conclusion

$$M(\mathbb{C}[x])^{M_0(\mathbb{C}[x])} = \mathbb{C}[x],$$

where $M(\mathbb{C}[x]) = \mathbb{C}\langle t_i | i \in \mathbb{Z}_+\rangle$ and $M_0(\mathbb{C}[x]) = \mathbb{C}\langle \bar{t}_i | i \in \mathbb{N}\rangle$ as explained in Section 3.2. The projection is $\pi(t_i) = \bar{t}_i$ for $i > 0$ and $\pi(t_0) = 0$. The induced coaction is therefore

$$\Delta_L t_0 = 1\otimes t_0, \qquad \Delta_L t_i = \sum_{j=1}^{j=i} \sum_{n_1+\cdots+n_j=i} \bar{t}_{n_1}\cdots\bar{t}_{n_j}\otimes t_j,$$

where all indices shown are in $\mathbb{N}$. Note that although the coproduct of $M(\mathbb{C}[x])$ involves formal powerseries, its product structure and the induced coaction are algebraic. Since $M(\mathbb{C}[x])$ and $M_0(\mathbb{C}[x])$ are free algebras, the fixed point subalgebra is $\mathbb{C}[t_0]$.

To go further, one needs to have versions of M, M_0 which are actually Hopf algebras and not merely bialgebras. This can typically be done by localizing, i.e. by adjoining suitable inverses or powerseries. For example, we may take the same M as for the two-point set but in the form of the comeasuring bialgebra for $\mathbb{C}[x]/x^2 = 1$ given by (14) in Section 3.3 (generated by $1, b, t$). For the splitting associated with this basis we have $M_0 = \mathbb{C}[\bar{t}]/\bar{t}^2 = 1$ which is actually a Hopf algebra. The map π is $\pi(b) = 0$ and $\pi(t) = \bar{t}$. The induced coaction is

$$\Delta_L b = 1\otimes b, \qquad \Delta_L t = \bar{t}\otimes t.$$

Since $t^2 = 1 - b^2$ in M, every element of that can be written in the form $f(b) + tg(b)$, of which the fixed elements are those with $g = 0$. Therefore the fixed subalgebra A, the base of the bundle, is

$$M^{M_0} = \mathbb{C}[b].$$

Note that the base is no longer the original algebra of which we took the 'diffeomorphisms'. (This is attributable to the nontrivial ${c_{ij}}^0$ in Proposition 2.3 for this basis). On the other hand, if we allow formal powerseries in b^2 then M has an antipode and, moreover, the nondegeneracy (Hopf–Galois) condition holds so that we have, at least formally, a homogeneous quantum principal bundle. The canonical form θ is also formal, for the same reason. Here $\ker\epsilon = \{f(b) + tg(b) \mid f(0) + g(0) = 0\}$ and hence V consists of polynomials in b vanishing at 0. Then

$$\theta(b) = b \otimes \frac{t}{1-b^2} - 1 \otimes \frac{bt}{1-b^2}.$$

On the other hand, the map $\Phi(\bar{t}) = t$ is a coalgebra map splitting of π and makes this localized bundle trivial. It should therefore be viewed only as a local 'patch' of a more nontrivial bundle without such localization. (This requires, a suitable extension of the theory in [5,6].) At least in this patch, a connection ω is induced as in [5] by a 'gauge field' $\alpha : M_0 \to \Omega^1\mathbb{C}[b]$ such that $\alpha(1) = 0$, which in our case means a single element $\alpha = \alpha(\bar{t}) \in \Omega^1\mathbb{C}[b]$. This induces the covariant derivative

$$\nabla : \Omega^1\mathbb{C}[b] \to \Omega^1\mathbb{C}[b] \otimes_{\mathbb{C}[b]} \Omega^1\mathbb{C}[b], \qquad \nabla\, \mathrm{d}b = (b \otimes 1 - 1 \otimes b)\alpha.$$

3.6. Preserving nonuniversal differential calculi

Here we discuss briefly the further restrictions imposed by nonuniversal differential calculi. We recall that for any unital algebra A, the universal calculus $\Omega^1 A$ is the kernel of the product map $A \otimes A \to A$ as an A-bimodule, while a general differential calculus is a quotient $\Omega^1(A) = \Omega^1 A/N$ for some sub-bimodule N. The differential is the universal one $\mathrm{d}a = a \otimes 1 - 1 \otimes a$ (as above) projected down to $\Omega^1(A)$ in the nonuniversal case. It is natural to restrict the notion of unital comeasurings to ones that preserve N in the sense

$$\beta(N) \subseteq N \otimes B, \tag{25}$$

where β is extended to the tensor square $A \otimes A$ in the usual way, restricted to $\Omega^1 A$. The universal object in this case is a quotient of $M(A)$. It is easy to see that it remains a bialgebra, which we denote $M(A, \Omega^1(A))$, the *comeasuring bialgebra with nonuniversal calculus*.

Thus, when $A = \mathbb{C}(\Sigma)$, the sub-bimodules N are classified by subsets of $\Sigma \times \Sigma - \mathrm{diag}$. In [9] the complement of this subset is denoted by E and we write $i - j$ when $(i, j) \in E$, and $i\#j$ when (i, j) is in the complement in $\Sigma \times \Sigma - \mathrm{diag}$. Thus, $N = \{\delta_i \otimes \delta_j \mid i\#j\}$. To describe the comeasuring bialgebra $M(\mathbb{C}(\Sigma), \Omega^1(\mathbb{C}(\Sigma)))$ in this case we use the base-point free version of $M(\mathbb{C}(\Sigma))$ defined by (17) and (19), and quotient further by the relations

$$\tau^i{}_j\tau^k{}_l = 0 \quad \forall i - k, \ \ j\#l \tag{26}$$

as the requirement that N is preserved by $\delta_j \mapsto \delta_i \otimes \tau^i{}_j$. Note that in $M(\mathbb{C}(\Sigma))$ we have, for $i - k$ and $j\#l$,

$$\Delta(\tau^i{}_j\tau^k{}_l) = \sum_{a-b} \tau^i{}_a\tau^k{}_b \otimes \tau^a{}_j\tau^b{}_l + \sum_{a\#b} \tau^i{}_a\tau^k{}_b \otimes \tau^a{}_j\tau^b{}_l.$$

Hence it is clear that the quotient by relations (26) remains a bialgebra. The case $a = b$ does not contribute here in view of $j \neq l$ and (17).

When Σ is the two-point set, there is (up to equivalence) only one nonuniversal nontrivial differential calculus, namely $E = \{(1,2)\}$. Then $M(\mathbb{C}(\Sigma), \Omega^1(\mathbb{C}(\Sigma)))$ is the quotient of the bialgebra (23) and (24) by the additional relation

$$(1-g_+)(1-g_-)=0.$$

Equivalently, we quotient the comeasuring bialgebra M for $\mathbb{C}[x]/x^2 = 1$, given in (14) in Section 3.3, by

$$t^2 = t(1+b).$$

When $A = \mathbb{C}[x]$ the natural differential calculi are those bicovariant under the coaddition structure, and are labelled by a single parameter $\lambda \in \mathbb{C}$ (and over a general field, the coirreducible calculi correspond to field extensions). The standard commutative differential calculus is the one where N is the subbimodule generated by $x\,\mathrm{d}x - (\mathrm{d}x)x$. This contains in particular all elements of the form $x^i\,\mathrm{d}x^j - (\mathrm{d}x^j)x^i$ for $i, j \in \mathbb{Z}_+$. Under coaction (4) we have

$$x\,\mathrm{d}x - (\mathrm{d}x)x \mapsto \sum_{ij}(x^i\,\mathrm{d}x^j - (\mathrm{d}x^j)x^i)\otimes t_i t_j + \sum(\mathrm{d}x^j)x^i \otimes (t_i t_j - t_j t_i).$$

Since the $(\mathrm{d}x^j)x^i$ are linearly independent elements of $\Omega^1\mathbb{C}[x]$ for $j > 0$, the differentiable comeasurings $M(\mathbb{C}[x], \Omega^1(\mathbb{C}[x]))$ for the standard commutative differential calculi are precisely the quotient of $M(\mathbb{C}[x]) = \mathbb{C}\langle t_i \,|\, i \in \mathbb{Z}_+\rangle$ by the relations of commutativity of the generators, i.e. precisely the usual $\mathrm{Diff}(\mathbb{C}[x])$. Differential calculi in between the universal one and the commutative one therefore lead to quotients in between the free comeasuring bialgebra $M(\mathbb{C}[x])$ and the classical commutative one.

3.7. Diffeomorphisms of the quantum-braided plane

Until now, we have only considered diffeomorphisms of the line and its quotients. We now move on to the plane, which is now nontrivial enough to have a noncommutative q-deformed version, the quantum-braided plane $\mathbb{C}_q^2$. This has generators $1, x, y$ with relations $yx = qxy$, where $q \neq 0$ is a parameter. It is infinite-dimensional but, as in Section 3.1, we find that $M_0(\mathbb{C}_q^2)$ is algebraic without the need for formal powerseries.

The computation of the comeasuring bialgebra $M(\mathbb{C}[x, y])$ for the classical plane follows just the same steps as in Section 3.1. With the basis $\{e_{m,n} = x^m y^n \mid m, n \in \mathbb{Z}_+\}$ one has the structure constants

$$c_{(i,j)(k,l)}{}^{(m,n)} = \delta^m_{i+k}\delta^n_{j+l}$$

and $M_1(\mathbb{C}[x, y])$ is generated by $1, t^{(i,j)}{}_{(k,l)}$ with relations

$$\sum_{(a,b)+(c,d)=(m,n)} t^{(a,b)}{}_{(i,j)}t^{(c,d)}{}_{(k,l)} = t^{(m,n)}{}_{(i,j)+(k,l)} \tag{27}$$

and the matrix coalgebra. Similarly, to Section 3.1, we find $M(\mathbb{C}[x,y])$ is generated by 1, $s_{(i,j)}, t_{(i,j)}$ where $i, j \in \mathbb{Z}_+$ modulo the relations

$$\sum_{(a,b)+(c,d)=(i,j)} s_{(a,b)}t_{(c,d)} = \sum_{(a,b)+(c,d)=(i,j)} t_{(a,b)}s_{(c,d)} \quad \forall i, j \in \mathbb{Z}_+. \tag{28}$$

Here

$$s_{(i,j)} \equiv t^{(i,j)}{}_{(1,0)}, \qquad t_{(i,j)} \equiv t^{(i,j)}{}_{(0,1)} \tag{29}$$

generate the others by repeated 'convolution'. Thus,

$$t^{(i,j)}{}_{(k,0)} = \sum_{(a_1,b_1)+\cdots(a_k,b_k)=(i,j)} s_{(a_1,b_1)}\cdots s_{(a_k,b_k)},$$

$$t^{(i,j)}{}_{(0,k)} = \sum_{(a_1,b_1)+\cdots(a_k,b_k)=(i,j)} t_{(a_1,b_1)}\cdots t_{(a_k,b_k)}$$

$$t^{(i,j)}{}_{(k,l)} = \sum_{(a,b)+(c,d)=(i,j)} t^{(a,b)}{}_{(k,0)}t^{(c,d)}{}_{(0,l)}$$

for $k, l \geq 1$. These follow from (27), while (28) is the residual content of (27) after making these substitutions. If we consider the $s_{(i,j)}, t_{(i,j)}$ as sequences on $\mathbb{Z}_+ \times \mathbb{Z}_+$ with convolution product $s * t_{(i,j)} = \sum_{(a,b)+(c,d)=(i,j)} s_{(a,b)}t_{(c,d)}$, we can write the residual relations of $M(\mathbb{C}[x,y])$ compactly as $s * t = t * s$. The coalgebra is

$$\Delta s_{(i,j)} = \sum_{(a,b)} t^{(i,j)}{}_{(a,b)} \otimes s_{(a,b)}, \qquad \Delta t_{(i,j)} = \sum_{(a,b)} t^{(i,j)}{}_{(a,b)} \otimes t_{(a,b)},$$
$$\epsilon(s_{(i,j)}) = \delta^i_1\delta^j_0, \qquad \epsilon(t_{(i,j)}) = \delta^i_0\delta^j_1. \tag{30}$$

The coaction on $\mathbb{C}[x,y]$ is

$$x \mapsto \sum_{(i,j)} x^i y^j \otimes s_{(i,j)}, \qquad y \mapsto \sum_{(i,j)} x^i y^j \otimes t_{(i,j)}. \tag{31}$$

The quotient $M_0(\mathbb{C}[x,y])$ has the same form but with the indices in (29)–(31) excluding (0,0). In this case $t^{(i,j)}{}_{(k,l)} = 0$ unless $i + j \geq k + l$, so that the coproduct in this case is a finite sum. The geometric meaning of $M_0(\mathbb{C}[x,y])$ is the diffeomorphisms that fix the origin.

For the quantum-braided plane, we again have a basis $\{e_{m,n} = x^m y^n \mid m, n \in \mathbb{Z}_+\}$, but now with the q-deformed structure constants and consequent relations

$$c_{(i,j)(k,l)}{}^{(m,n)} = \delta^m_{i+k}\delta^n_{j+l}q^{jk},$$
$$\sum_{(a,b)+(c,d)=(m,n)} q^{bc}t^{(a,b)}{}_{(i,j)}t^{(c,d)}{}_{(k,l)} = q^{jk}t^{(m,n)}{}_{(ij)+(k,l)}. \tag{32}$$

Proposition 3.1. *The comeasuring bialgebra $M(\mathbb{C}_q^2)$ of the quantum-braided plane is generated by 1, $s_{(i,j)}$, $t_{(i,j)}$ for $i, j \in \mathbb{Z}_+$, modulo the relations*

$$q \sum_{(a,b)+(c,d)=(i,j)} q^{bc} s_{(a,b)} t_{(c,d)} = \sum_{(a,b)+(c,d)=(i,j)} q^{bc} t_{(a,b)} s_{(c,d)} \quad \forall i, j \in \mathbb{Z}_+.$$

The coaction is as in (31). The higher generators and hence the coproduct (30) are given by

$$t^{(i,j)}{}_{(k,0)} = \sum_{(a_1,b_1)+\cdots+(a_k,b_k)=(i,j)} s_{(a_1,b_1)} \cdots s_{(a_k,b_k)} q^{\sum_{s=2}^{k}(b_1+\cdots+b_{s-1})a_s},$$

$$t^{(i,j)}{}_{(0,k)} = \sum_{(a_1,b_1)+\cdots+(a_k,b_k)=(i,j)} t_{(a_1,b_1)} \cdots t_{(a_k,b_k)} q^{\sum_{s=2}^{k}(b_1+\cdots+b_{s-1})a_s},$$

$$t^{(i,j)}{}_{(k,l)} = \sum_{(a,b)+(c,d)=(i,j)} q^{bc} t^{(a,b)}{}_{(k,0)} t^{(c,d)}{}_{(0,l)}$$

for $k \geq 1$.

Proof. Relations (32) allow one to define the general $t^{(i,j)}{}_{(k,l)}$ as stated in terms of the $s_{(i,j)}, t_{(i,j)}$. Putting these back into (32) leaves the residual relations between s, t as shown. These are obtained from $t^{(i,j)}_{(1,0)+(0,1)} = t^{(i,j)}_{(1,1)} = t^{(i,j)}_{(0,1)+(1,0)}$ computed from (32), i.e. they reflect commutativity of the addition law on $\mathbb{Z}_+ \times \mathbb{Z}_+$. To see that these are all the relations, it is useful to define the q-deformed convolution product $s *_q t_q t_{(i,j)} = \sum_{(a,b)+(c,d)=(i,j)} q^{bc} s_{(a,b)} t_{(c,d)}$ for sequences on $\mathbb{Z}_+ \times \mathbb{Z}_+$. One may check that $*_q$ is associative. Then $t^{(i,j)}{}_{(k,0)} = (s *_q \cdots *_q s)_{(i,j)}$ and $t^{(i,j)}{}_{(0,k)} = (t *_q \cdots *_q t)_{(i,j)}$ (k-fold products) and $t^{(i,j)}{}_{(k,l)} = (s *_q \cdots *_q s *_q t *_q \cdots *_q t)_{(i,j)}$ (k-fold and l-fold). Relations (32) then hold if $t *_q s = qs *_q t$, which are the stated relations of $M(\mathbb{C}_q^2)$. □

The quotient $M_0(\mathbb{C}_q^2)$ has the same form with the index (0,0) excluded from the expressions in the proposition. In this case $t^{(i,j)}{}_{(k,l)} = 0$ unless $i + j \geq k + l$, so that the coproduct is a finite sum. The lowest level generators of $M_0(\mathbb{C}_q^2)$ are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} s_{(1,0)} & t_{(1,0)} \\ s_{(0,1)} & t_{(0,1)} \end{pmatrix} = \begin{pmatrix} t^{(1,0)}{}_{(1,0)} & t^{(1,0)}{}_{(0,1)} \\ t^{(0,1)}{}_{(1,0)} & t^{(0,1)}{}_{(0,1)} \end{pmatrix} \tag{33}$$

and the relations among these are

$$dc = qcd, \qquad ba = qab, \qquad ad - da = q^{-1}bc - qcb,$$

which are just half of the relations of the quantum matrices. Moreover, this is just the lowest level content of the relations $t *_q s = qs *_q t$.

We may similarly consider comeasuring bialgebras for the fermionic quantum plane $\mathbb{C}_q^{0|2}$. The latter is a four-dimensional algebra generated by $1, \theta, \vartheta$ with relations $\vartheta\theta = -q^{-1}\theta\vartheta$ and $\theta^2 = \vartheta^2 = 0$. It can be viewed geometrically as the natural algebra of exact differentials

$\theta = \mathrm{d}x$, $\vartheta = \mathrm{d}y$ on the bosonic quantum plane $\mathbb{C}_q^2$. As basis we take $e_0 = 1$, $e_1 = \theta$, $e_2 = \vartheta$ and $e_3 = \theta\vartheta$. The only possibly nonzero structure constants are clearly

$$c_{0i}{}^j = \delta_i^j = c_{i0}{}^j, \qquad c_{12}{}^3 = 1, \qquad c_{21}{}^3 = -q^{-1}.$$

Proposition 3.2. *The comeasuring bialgebra $M(\mathbb{C}_q^{0|2})$ of the fermionic quantum-braided plane is generated by 1, b_1, b_2, a, b, c, d, α, β, modulo the relations*

$$\begin{aligned}
&b_1^2 = b_2^2 = 0, \qquad \{b_1, b_2\}_q = 0,\\
&\{b_1, a\} = \{b_1, c\} = \{b_2, b\} = \{b_2, d\} = 0,\\
&\{b_1, b\}_q + \{a, b_2\}_q = 0, \qquad \{b_1, d\}_q + \{c, b_2\}_q = 0,\\
&ac - q^{-1}ca + \{b_1, \alpha\} = 0, \qquad bd - q^{-1}db + \{b_2, \beta\} = 0,\\
&ad - da + \{b_1, \beta\}_q + \{\alpha, b_2\}_q = q^{-1}cb - qbc,
\end{aligned}$$

where $\{\ ,\ \}$ denotes anticommutator and $\{b_1, b_2\}_q = b_1b_2 + qb_2b_1$, etc. The full b_i, $t^i{}_j$ and hence the coalgebra from Proposition 2.3 are given by

$$b_3 = b_1b_2, \qquad \mathbf{t} = \begin{pmatrix} a & b & b_1b + ab_2 \\ c & d & b_1d + cb_2 \\ \alpha & \beta & ad - q^{-1}cb + b_1\beta + \alpha b_2 \end{pmatrix}.$$

Proof. This is a direct computation from Proposition 2.3. Here $c_{ij}{}^0 = 0$ for $i, j \neq 0$, so that $c_{12}{}^3 b_3 = b_1b_2$ and $c_{21}{}^3 b_3 = b_2b_1$ tell us b_3 and that $b_1b_2 = -qb_2b_1$, etc. □

We see that the translation generators b_1, b_2 themselves form a fermionic quantum plane $\mathbb{C}_q^{0|2}$. Setting them to zero gives $M_0(\mathbb{C}_q^{0|2})$ as generated by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

modulo the relations

$$ca = qac, \qquad db = qbd, \qquad ad - da = q^{-1}cb - qbc$$

and no relations involving α, β, which are free. Note that these are the 'other half' of the standard 2×2 quantum matrix generators (in contrast to the lowest level of $M_0(\mathbb{C}_q^2)$ given above), a situation somewhat similar to [7]. In our case it means that one may identify the standard quantum matrices geometrically as the lowest level of the diffeomorphisms that preserve the entire exterior algebra on $\mathbb{C}_q^2$.

3.8. Preserving a coaddition

We are now ready to consider the general theory of restricting diffeomorphisms to those preserving a coalgebra structure. In our 'coordinate ring' setting it means a quotient of $M_1(A)$, which we denote $M_1(A, \Delta)$. Thus, a coalgebra structure on A corresponds in a basis $\{e_i\}$ to structure constants defined by $\Delta e_i = d_i{}^{jk} e_j \otimes e_k$. Along the same lines as in Section 2, the comeasurings that preserve this clearly means the quotient of $M_1(A)$ by the additional relation

$$d_a{}^{jk} t^a{}_i = d_i{}^{ab} t^j{}_a t^k{}_b. \tag{34}$$

As explained below (Proposition 2.2), one has a bialgebra for any $d_i{}^{jk}$, i.e. one does not need Δ to be coassociative or to make A into a bialgebra. The coassociativity is, however, natural to assume.

We also have quotients $M(A, \Delta)$, $M_0(A, \Delta)$, etc., as before. And in principle, we have still further quotients where a counit ϵ is respected as well. However, the counit in the case of a bialgebra A defines a natural splitting $A = 1 \oplus A'$, where $A' = \ker \epsilon$, i.e. it is natural to choose the basis so that $\epsilon(1) = 1$ and $\epsilon(e_i) = 0$ for $i > 0$. At least in this case, the bialgebra $M_0(A, \Delta)$ automatically preserves the counit without further quotients arising from that.

The quantum-braided plane $\mathbb{C}_q^2$ has such a coalgebra structure, expressing the braided addition law. This is the braided 'coaddition' introduced in [10]. Explicitly, it is given by

$$\Delta_+(x^m y^n) = \sum_{r=0}^{m} \sum_{s=0}^{n} \begin{bmatrix} m \\ r \end{bmatrix}_{q^2} \begin{bmatrix} n \\ s \end{bmatrix}_{q^2} x^r y^s \otimes x^{m-r} y^{n-s} q^{(m-r)s},$$

where we use the q^2-binomial coefficients defined as usual, but in terms of q^2-factorials

$$[n]_{q^2}! = [n]_{q^2} \cdots [1]_{q^2}, \qquad [m]_{q^2} = \frac{1 - q^{2m}}{1 - q^2}.$$

Although this coproduct forms a coalgebra (it is coassociative), it does not make $\mathbb{C}_q^2$ into a usual bialgebra or Hopf algebra, but rather into a braided group [11]. We will say more about this in Section 4; for the present purposes we need only to know its explicit form as stated here. For our above basis, we have

$$d_{(m,n)}{}^{(i,j)(k,l)} = \delta_m^{i+k} \delta_n^{j+l} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} q^{jk}.$$

Proposition 3.3. *The quotient of the restricted comeasuring bialgebra $M_0(\mathbb{C}_q^2)$ respecting the braided coaddition on $\mathbb{C}_q^2$, for generic q, can be identified with the standard 2×2 quantum matrices $M_q(2)$.*

Proof. The additional quotient of $M_0(\mathbb{C}_q^2)$ is by the relation

$$\sum_{(a,b)+(c,d)=(m,n)} \begin{bmatrix} m \\ a \end{bmatrix}_{q^2} \begin{bmatrix} n \\ b \end{bmatrix}_{q^2} q^{bc} t^{(i,j)}{}_{(a,b)} t^{(k,l)}{}_{(c,d)}$$

$$= \begin{bmatrix} i+k \\ i \end{bmatrix}_{q^2} \begin{bmatrix} j+l \\ j \end{bmatrix}_{q^2} q^{jk} t^{(i,j)+(k,l)}{}_{(m,n)}.$$

For generic q we can define new generators

$$\tau^{(i,j)}{}_{(k,l)} = t^{(i,j)}{}_{(k,l)} \frac{[i]_{q^2}![j]_{q^2}!}{[k]_{q^2}![l]_{q^2}!}$$

and then the additional relations become

$$\sum_{(a,b)+(c,d)=(m,n)} q^{bc} \tau^{(i,j)}{}_{(a,b)} \tau^{(k,l)}{}_{(c,d)} = q^{jk} \tau^{(i,j)+(k,l)}{}_{(m,n)},$$

which, by similar reasoning as for the comeasuring bialgebra, implies that the generators can be obtained by convolution from the generators $\sigma_{(i,j)} = \tau^{(1,0)}{}_{(i,j)}$ and $\tau_{(i,j)} = \tau^{(0,1)}{}_{(i,j)}$. This time

$$\tau^{(k,0)}{}_{(i,j)} = \sum_{(a_1,b_1)+\cdots(a_k,b_k)=(i,j)} \sigma_{(a_1,b_1)} \cdots \sigma_{(a_k,b_k)} q^{\sum_{s=2}^{k}(b_1+\cdots+b_{s-1})a_s},$$

$$\tau^{(0,k)}{}_{(i,j)} = \sum_{(a_1,b_1)+\cdots(a_k,b_k)=(i,j)} \tau_{(a_1,b_1)} \cdots \tau_{(a_k,b_k)} q^{\sum_{s=2}^{k}(b_1+\cdots+b_{s-1})a_s},$$

$$\tau^{(k,l)}{}_{(i,j)} = \sum_{(a,b)+(c,d)=(i,j)} q^{bc} \tau^{(k,0)}{}_{(a,b)} \tau^{(0,l)}{}_{(c,d)}$$

for $k, l \geq 1$. The residual relations are

$$q \sum_{(a,b)+(c,d)=(i,j)} q^{bc} \sigma_{(a,b)} \tau_{(c,d)} = \sum_{(a,b)+(c,d)=(i,j)} q^{bc} \tau_{(a,b)} \sigma_{(c,d)}, \quad \forall i, j \in \mathbb{Z}_+$$

or, in the convolution notation, $\tau *_q \sigma = q\sigma *_q \tau$. This is the bialgebra respecting only the coalgebra Δ_+ (indeed, the quantum-braided plane is self-dual as a braided group and this is why the algebra has the same form as the comeasuring bialgebra).

These convolution formulae imply that $t^{(i,j)}{}_{(k,l)} = 0$ unless $i + j \leq k + l$. Combined with the reverse inequality for $M_0(\mathbb{C}_q^2)$, we see that $t^{(i,j)}{}_{(k,l)} = 0$ in $M_0(\mathbb{C}_q^2, \Delta_+)$ unless $i + j = k + l$. It follows that $M_0(\mathbb{C}_q^2, \Delta_+)$ is generated by 1 and the lowest level generators (33). Half their relations are given above, inherited from $M_0(\mathbb{C}_q^2)$. The other half come from the relations $\tau *_q \sigma = q\sigma *_q \tau$, which are computed similarly as

$$db = qbd, \quad ca = qac, \quad ad - da = q^{-1}cb - qbc.$$

Thus, $M_0(\mathbb{C}_q^2, \Delta_+) = M_q(2)$, the standard 2×2 quantum matrices in the conventions of [8]. The coaction reduces to the standard coaction on $\mathbb{C}_q^2$. □

4. R-matrix constructions for comeasuring bialgebras

In this section we consider some general constructions possible when our algebra is braided, i.e. in the presence of a Yang–Baxter operator. As a first application, we note that until now we have studied the maximal comeasuring objects in the category of bialgebras. Since they are maximal they tend to be free, with only the minimal relations compatible with coacting on our algebra. However, when the algebra is itself braided, we can look for objects maximal in some braided-commutative sense. We then give more functorial braided group versions of these constructions, related by transmutation.

We recall that an algebra A is braided if it comes with an operator $\Psi : A\otimes A \to A\otimes A$ obeying the braid relations and functorial with respect to the product of A in the sense

$$\begin{aligned}\Psi(ab\otimes c) &= (\mathrm{id}\otimes \cdot)(\Psi\otimes\mathrm{id})(\mathrm{id}\otimes\Psi)(a\otimes b\otimes c),\\ \Psi(a\otimes bc) &= (\cdot\otimes\mathrm{id})(\mathrm{id}\otimes\Psi)(\Psi\otimes\mathrm{id})(a\otimes b\otimes c),\end{aligned}\tag{35}$$

for all $a, b, c \in A$. More generally, a braided algebra means A an object in a braided category (such as that generated by a single braiding operator) with the product a morphism. We recall that in a braided category there is a braiding between any two objects playing the role of 'transposition'. Also, the (co)modules of any (dual) quasitriangular bialgebra or Hopf algebra (i.e., of any 'strict' quantum group) form a braided category, so any algebra covariant under a strict quantum group is a braided algebra. See [8] or papers such as [11,12] where braided algebras and groups have been introduced. In this setting we introduce $M_1(R, A)$ as again a dual quasitriangular bialgebra or 'strict' quantum group.

Thus, when A has a basis $\{e_i\}$, a braided algebra translates into the existence of a matrix $R\in M_n\otimes M_n$ (where M_n denotes $n\times n$-matrices and $n=\dim(A)$) obeying the Quantum Yang–Baxter Equations (QYBE) $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ (a so-called R-matrix), such that

$$c_{12}{}^{3}R_{14}R_{24} = R_{34}c_{12}{}^{3},\qquad R_{12}c_{34}{}^{2} = c_{34}{}^{2}R_{14}R_{13}.\tag{36}$$

We use the standard compact notation where $R_{12} = R\otimes\mathrm{id}$ and we suppose that R is invertible. In explicit component terms, the requirement is

$$\begin{aligned}c_{ij}{}^{a}R^{k}{}_{a}{}^{m}{}_{n} &= c_{ab}{}^{k}R^{a}{}_{i}{}^{m}{}_{c}R^{b}{}_{j}{}^{c}{}_{n},\\ c_{jk}{}^{a}R^{m}{}_{n}{}^{i}{}_{a} &= c_{ab}{}^{i}R^{m}{}_{c}{}^{b}{}_{k}R^{c}{}_{n}{}^{a}{}_{j},\end{aligned}\tag{37}$$

where

$$\Psi(e_i\otimes e_j) = e_b\otimes e_a R^{a}{}_{i}{}^{b}{}_{j}.$$

4.1. Dual quasi-triangular comeasuring bialgebras

We also recall that a bialgebra M is dual quasi-triangular or $\mathcal{R}$-commutative if there exists $\mathcal{R} : M\otimes M\to k$ such that

$$\mathcal{R}(ab, c) = \mathcal{R}(a, c_{(1)})\mathcal{R}(b, c_{(2)}),$$

$$\mathcal{R}(a, bc) = \mathcal{R}(a_{(1)}, c)\mathcal{R}(a_{(2)}, b),$$

$$b_{(1)}a_{(1)}\mathcal{R}(a_{(2)}, b_{(2)}) = \mathcal{R}(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}$$

for all $a, b, c \in M$. The axioms are dual to the quasi-triangular structures introduced for the quantum groups $U_q(g)$ by Drinfeld [13]. On the other hand, given any R-matrix there is a dual quasi-triangular bialgebra $A(R)$ of 'quantum matrices' with generators $R\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1 R$ (in a compact notation where $\mathbf{t}_1 = \mathbf{t} \otimes \mathrm{id}$, etc.). These are the FRT relations [14] while the dual quasi-triangularity for general R is due to the author [15] and takes the form $\mathcal{R}(t^i{}_j, t^k{}_l) = R^i{}_j{}^k{}_l$. See [8].

Proposition 4.1. *If A is a braided algebra with braiding defined by an R-matrix, then $M_1(R, A)$ defined by the matrix coalgebra and the relations*

$$R^i{}_a{}^j{}_b t^a{}_k t^b{}_l = t^j{}_b t^i{}_a R^a{}_k{}^b{}_l, \qquad c_{ij}{}^a t^k{}_a = c_{ab}{}^k t^a{}_i t^b{}_j$$

is a dual quasi-triangular bialgebra.

Proof. We check that this quotient of $A(R)$, in which the relations of Proposition 2.2 are further imposed, inherits the linear functional $\mathcal{R}$. If so then it will remain a dual quasi-triangular structure for the quotient. Thus,

$$\mathcal{R}(t^i{}_j, c_{ab}{}^k t^a{}_m t^b{}_n) = R^i{}_c{}^b{}_n R^c{}_j{}^a{}_m c_{ab}{}^k = c_{mn}{}^a R^i{}_j{}^k{}_a = \mathcal{R}(t^i{}_j, c_{mn}{}^a t^k{}_a),$$

$$\mathcal{R}(c_{ab}{}^k t^a{}_i t^b{}_j, t^m{}_n) = c^k_{ab} R^a{}_i{}^m{}_c R^b{}_j{}^c{}_n = c_{ij}{}^a R^k{}_a{}^m{}_n = \mathcal{R}(c_{ij}{}^a t^k{}_a, t^m{}_n)$$

using the covariance conditions (37). The proof for $t^i{}_j$ and $t^m{}_n$ replaced by general strings of generators has just the same form, with repeated use of the covariance conditions; the general proof thereby proceeds by a straightforward induction. □

While this quotient will always be dual quasi-triangular, it may also be trivial, i.e. the ideal generated by both sets of relations may be too large. Although not necessary, if A is itself 'braided commutative' in some sense then one may expect that the above construction is more natural. The appropriate form of commutativity is, as for braided matrices and braided planes [10,12], the existence of a matrix $R' \in M_n \otimes M_n$ obeying

$$R'_{12}R_{13}R_{23} = R_{23}R_{13}R'_{12}, \qquad R_{12}R_{13}R'_{23} = R'_{23}R_{13}R_{12}. \tag{38}$$

More precisely, these matrix conditions follow from and are essentially equivalent to the algebraic condition $R'\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1 R'$ in the quantum matrix algebra. Typically, R' is built from R and the relations with R' are equivalent to the defining relations with R. With respect to this, braided-commutativity is

$$c_{ij}{}^k = c_{ba}{}^k R'^a{}_i{}^b{}_j. \tag{39}$$

Note that the more naive definition of braided commutativity would be $\cdot \circ \Psi(a \otimes b) = ab$ for all a, b, but this is only natural when $\Psi^2 = \mathrm{id}$. It corresponds to the choice $R' = R$, which is too restrictive to apply in most examples of q-deformation. Then

$$c_{ba}{}^k(t^b{}_jt^a{}_iR'^i{}_m{}^j{}_n) = c_{ba}{}^kR'^a{}_i{}^b{}_jt^i{}_mt^j{}_n = c_{ij}{}^kt^i{}_mt^j{}_n = c_{mn}{}^at^k{}_a$$
$$= c_{ji}{}^at^k{}_aR'^i{}_m{}^j{}_n = (c_{ba}{}^kt^b{}_jt^a{}_i)R'^i{}_m{}^j{}_n$$

holds automatically in $M_1(R, A)$.

As before, there are quotients $M(R, A)$ and $M_0(R, A)$ when Ψ respects $e_0 = 1$. The natural condition is $\Psi(a \otimes 1) = 1 \otimes a$ and $\Psi(1 \otimes a) = a \otimes 1$ for all $a \in A$.

As a finite-dimensional example, it is shown in [4] that the commutative quotient of the comeasuring bialgebra M_0 for the quaternions recovers the coordinate algebra of SO_3. All the ingredients here q-deform, i.e. it is clear that one may recover $M_0(R, \mathbb{H}_q) = SO_q(3)$ in the similar way, where R is an extension of the so_3 type R-matrix and the q-quaternions are built from the known $SO_q(3)$-covariant q-epsilon tensor and q-metric. Details will be presented elsewhere.

For the simplest 'geometrical' example, namely $A = \mathbb{C}[x]$, we have more than one way to consider it as a braided algebra. The simplest is as the braided line [10,16] where

$$\Psi(x^i \otimes x^j) = q^{ij}x^j \otimes x^i, \qquad R^i{}_j{}^k{}_l = \delta^i{}_j\delta^k{}_lq^{ik}. \tag{40}$$

Then $M(R, \mathbb{C}[x]) = \mathrm{Diff}(\mathbb{C}[x])$ is the classical commutative diffeomorphism group. Here q cancels from both sides and the result is the same as for $q = 1$, i.e. the imposition of commutativity between all the generators.

A different braiding on $A = \mathbb{C}[x]$ is the one introduced in [16],

$$\Psi(x^i \otimes x^j) = \sum_{k=0}^{i} \begin{bmatrix} i \\ k \end{bmatrix}_q q^{j(i-k)}(1-q)^k \times \frac{[j+k-1]_q!}{[j-1]_q!}x^{j+k} \otimes x^{i-k} \quad \forall i, j \in \mathbb{N} \tag{41}$$

and the trivial transposition when i or $j = 0$. This is the canonical 'double' braiding associated to any braided group, in this case the braided line. The algebra of the braided group is automatically a braided algebra under its canonical braiding. From another point of view, the double bosonisation of a braided group canonically acts on the braided group. Here the double bosonisation of the braided line is $U_q(su_2)$ and acts as a q-deformation of $so(2, 1)$ by 'conformal transformation' on $\mathbb{C}[x]$ [17]. The above braiding is the one induced from the quasi-triangular structure of $U_q(su_2)$ by this action. We therefore call this the 'conformally braided' line.

Proposition 4.2. *The restricted dual-quasi-triangular comeasuring bialgebra $M_0(R, \mathbb{C}[x])$ of the conformally braided line is generated by 1, t_i for $i \in \mathbb{N}$ with the relations*

$$qt_jt_i = \sum_{k=0}^{j-1} \frac{[i+k]_q![j-1]_q!}{[j-k-1]_q[i]_q![k]_q!}q^{i(j-k)}(1-q)^kt_{i+k}t_{j-k}$$

and the coalgebra (5) as in Section 3.1. The dual quasi-triangular structure is $\mathcal{R}(t_i, t_j) = q\delta^i_1\delta^j_1$.

Proof. The R-matrix corresponding to (41) is

$$R^i{}_0{}^j{}_k=\delta^i{}_0\delta^j{}_k,\qquad R^i{}_j{}^k{}_0=\delta^i{}_j\delta^k{}_0,$$
$$R^i{}_j{}^k{}_l=\delta^{i+k}_{j+l}\begin{bmatrix} j\\ j-i\end{bmatrix}_q q^{li}(1-q)^{j-i}\frac{[k-1]_q!}{[l-1]_q!}$$

for $j,k\in\mathbb{N}$ and $i\le j$ in the third expression (which is zero otherwise), which implies that $R^a{}_1{}^b{}_1=q\delta^a_1\delta^b_1$ when $a,b\in\mathbb{N}$. Hence the $R\mathbf{t}_1\mathbf{t}_2=\mathbf{t}_2\mathbf{t}_1R$ relations on the generators $t_i\equiv t^i{}_1$ have the form

$$\sum_{a,b}R^i{}_a{}^j{}_a t_a t_b=qt_jt_i,$$

which then computes as stated. The R-matrix also provides the dual-quasi-triangular structure on the generators. □

In particular, the relations of the lowest order generators with the general generators are

$$t_1t_i=q^{i-1}t_it_1,\qquad t_2t_i=q^{2i-1}t_it_2+(1-q^{i+1})q^{i-1}t_{i+1}t_1,$$
$$t_3t_i=q^{3i-1}t_it_3+(1-q^{i+1})(1+q)q^{2i-1}t_{i+1}t_2$$
$$+(1-q^{i+2})(1-q^{i+1})q^{i-1}t_{i+2}t_1$$

for all $i\in\mathbb{N}$ etc. From this, for generic q, one obtains explicitly

$$t_1t_2=qt_2t_1,\qquad t_1t_3=q^2t_3t_1=q(t_2)^2,$$
$$t_2t_3=qt_3t_2=q^2t_4t_1,\qquad \Delta t_1=t_1\otimes t_1,\qquad \Delta t_2=t_2\otimes t_1+(t_1)^2\otimes t_2,$$
$$\Delta t_3=t_3\otimes t_1+(1+q)t_2t_1\otimes t_2+(t_1)^3\otimes t_3$$

and so on, to all orders. The comeasuring bialgebra $M(R,\mathbb{C}[x])$ is similar, with the extra generator t_0. The coproduct in this case involves infinite sums, so has to be treated formally.

For anyonic variables $x^3=0$, we again have a braided group (the anyonic line), and hence a canonical 'double' braiding. Here $q^3=1$ and the resulting 9×9 R-matrix in this case is given in [16, Ex. 4.8] as

$$R=\begin{pmatrix}
1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0\\
0&0&1&0&1-q&0&0&0&0\\
0&0&0&1&0&0&0&0&0\\
0&0&0&0&q&0&0&0&0\\
0&0&0&0&0&q^2&0&q-1&0\\
0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&q^2&0\\
0&0&0&0&0&0&0&0&q
\end{pmatrix}$$

as an endomorphism in basis $1\otimes 1, 1\otimes x, 1\otimes x^2, x\otimes 1,\dots,x^2\otimes x^2$. The comeasuring 'diffeomorphism' bialgebra $M(R,\mathbb{C}[x]/x^3)$ in this case is given by generators $1,b,s,t$

with the coalgebra (11) in Section 3.2, but now with the relations and dual-quasitriangular structure

$$b^3 = 0, \quad tb = qbt, \quad sb = qbs, \quad ts = qst,$$
$$\mathcal{R}(b,s) = 1 - q, \qquad \mathcal{R}(t,t) = q,$$

and $\mathcal{R}(b,b) = \mathcal{R}(b,t) = \mathcal{R}(t,b) = \mathcal{R}(t,s) = \mathcal{R}(s,b) = \mathcal{R}(s,t) = \mathcal{R}(s,s) = 0$. The variable b implements translation of the anyonic line, and we see that it is itself an anyonic variable. The quotient $M_0(R, \mathbb{C}[x]/x^3)$ is given by setting $b = 0$ and yields a dual-quasitriangular bialgebra with relations $ts = qst$ and the coalgebra structure (12) in Section 3.2.

Finally, we let $A = \mathbb{C}_q^2$ be the quantum-braided plane with generators x, y and relations $yx = qxy$, as in Section 3.7. This has a natural braiding Ψ used to describe coaddition on the quantum plane [10], as studied in Section 3.8. The quantum-braided plane is also braided commutative with respect to a certain matrix R'. Explicitly,

$$\Psi(x \otimes x) = q^2 x \otimes x, \qquad \Psi(y \otimes y) = q^2 y \otimes y,$$
$$\Psi(x \otimes y) = qy \otimes x, \qquad \Psi(y \otimes x) = qx \otimes y + (q^2 - 1) y \otimes x \tag{42}$$

extended to products by functoriality. This is the braiding induced by the action of $U_q(\tilde{s}u_2)$ on the quantum-braided plane and is usually given by an array of su_2-type R-matrices, see [8,10].

Proposition 4.3. *The dual-quasi-triangular comeasuring bialgebra $M(R, \mathbb{C}_q^2)$ is generated by 1, $s_{(i,j)}$, $t_{(i,j)}$, $i, j \in \mathbb{Z}_+$ with the relations in Proposition 3.1 and the additional relations*

$$R^{(i,j)}{}_{(a,b)}{}^{(k,l)}{}_{(c,d)} s_{(a,b)} s_{(c,d)} = q^2 s_{(k,l)} s_{(i,j)},$$
$$R^{(i,j)}{}_{(a,b)}{}^{(k,l)}{}_{(c,d)} t_{(a,b)} t_{(c,d)} = q^2 t_{(k,l)} t_{(i,j)},$$
$$R^{(i,j)}{}_{(a,b)}{}^{(k,l)}{}_{(c,d)} s_{(a,b)} t_{(c,d)} = q t_{(k,l)} s_{(i,j)},$$
$$R^{(i,j)}{}_{(a,b)}{}^{(k,l)}{}_{(c,d)} t_{(a,b)} s_{(c,d)} = q s_{(k,l)} t_{(i,j)} + (q^2 - 1) t_{(k,l)} s_{(i,j)},$$

where R corresponds to the braiding Ψ on $\mathbb{C}_q^2$. Summation over $a, b, c, d \in \mathbb{Z}_+$ should be understood.

Proof. We already know $M(\mathbb{C}_q^2)$ from Section 3.7, and now quotient this further by the $R\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1 R$ relations. On the other hand, the braiding on the generators (42) immediately gives $R^{(a,b)}{}_{(1,0)}{}^{(c,d)}{}_{(1,0)} = \delta^a_1 \delta^b_0 \delta^c_1 \delta^d_0 q^2$ corresponding to $\Psi(x \otimes x) = q^2 x \otimes x$, etc. Thus the additional relations on the generators $s_{(i,j)}, t_{(i,j)}$ reduce to the four sets as shown. They can be stated more compactly as the relations of a rectangular [18] quantum matrix $A(R : R_{su_2})$, where R_{su_2} is the standard su_2-type R-matrix. □

The relations of $M(R, \mathbb{C}_q^2)$ may be further expressed in terms of R_{su_2} or, alternatively, defined by induction. If $x_i = (x, y)$ is an $SU_q(2)$-covariant covector notation then

$$x_{i_1} x_{i_2} \cdots x_{i_n} = q^{p(i_1,\cdots,i_n)} x^{\#(i_1,\cdots,i_n)} y^{n-\#(i_1,\cdots,i_n)}$$

defines $\mathbb{Z}_+$-valued functions $p, \#$ (here $\#$ is the number of indices with value 1, and p is the number of indices with value 1 to the right of each index with value 2.) Then

$$R^{(a,b)}{}_{(i,j)}{}^{(c,d)}{}_{(k,l)}$$

$$= \delta^{a+b}_{i+j}\delta^{c+d}_{k+l}\sum_{\#(b_i)=c}\sum_{\#(a_i)=a} q^{p(b_i)+p(a_i)} Z^{a_1\cdots a_{i+j},b_1\ldots b_{k+l}}_{1\ldots 2\ldots,1\ldots 2\ldots},$$

where there are i indices $1\ldots$, j indices $2\ldots$, followed by k indices $1\ldots$, and l indices $2\ldots$, and where Z is the 'partition function' array of copies of R_{su_2} corresponding to Ψ in the braided covector description of the quantum-braided plane (see [8, Theorem 10.2.1]). Alternatively, the general Ψ in our basis $e_{(i,j)} = x^i y^j$ may be obtained by induction, via the formulae

$$\Psi(y^i\otimes x^j) = x\Psi(y^i\otimes x^{j-1})q^i$$
$$+(q^2-1)[i]_{q^2}q^{2(j-1)}y\Psi(y^{i-1}\otimes x^{j-1})x$$

and

$$\Psi(x^i y^j\otimes x^k y^l) = q^{i(2k+l-j)+l(2j-k)}y^l\Psi(y^j\otimes x^k)x^i.$$

These expressions follow from the functoriality (35) of the braiding.

Corollary 4.4. *There is a bialgebra surjection $M_0(R,\mathbb{C}) \to M_q(2)$ and $M_q(2)$ appears as a subalgebra (33) covered by this surjection. The dual-quasi-triangular structure of $M_0(R,\mathbb{C})$ extends that of $M_q(2)$.*

Proof. Since Ψ preserves the total degree, the only way to obtain a nonzero coefficient of $x\otimes x, x\otimes y, y\otimes x$ or $y\otimes y$ in $\Psi(x^i y^j\otimes x^k y^l)$ is with $i+j = 1 = k+l$. Hence to compute $R^{(1,0)}{}_{(i,j)}{}^{(1,0)}{}_{(k,l)}$ etc., we need only to consider (42). Writing $1 \equiv (1,0)$ and $2 = (0,1)$, the only nonzero entries of this form are given by the standard R_{su_2}. Thus, the lowest level generators $s_{(1,0)}, s_{(0,1)}$ obey the relations of the quantum-plane in R-matrix form, i.e. $ca = qac$ in the notation (33). Similarly, $t_{(1,0)}, t_{(0,1)}$ form a quantum-braided plane, i.e. $db = qbd$. These relations already hold in $M_0(\mathbb{C}_q^2)$. The third relation in Proposition 4.3 similarly reduces to the four relations $ba = qab$, $dc = qcd$, $cb = bc$ and $ad - da = (q^{-1}-q)bc$, while the fourth is then redundant. Hence the relations among these lowest level generators of $M_0(R,\mathbb{C}_q^2)$ are precisely the relations of the 2×2 quantum matrices $M_{\mathrm{q}}(2)$. □

The geometric meaning of this is as follows. The surjection corresponds classically to the inclusion of 2×2 linear transformations among the algebraic diffeomorphisms of the plane. The inclusion of $M_q(2)$ corresponds classically to the projection which associates to a diffeomorphism fixing zero its differential at zero, i.e. the linear transformation induced on the tangent space at zero.

Similalry, there is a surjection from $M(R,\mathbb{C}_q^2)$ to the q-deformed Weyl algebra $\mathbb{C}_q^2 \rtimes M_q(2)$ cf. [8] and an inclusion of it. One replaces the dilaton-extended $SU_q(2)$, i.e. $GL_q(2)$, in the bosonisation construction $\mathbb{C}_q^2 \rtimes\!\!\!\blacktriangleleft GL_q(2)$ from [8,10] by the quantum matrices $M_q(2)$. Conversely, at least these quotients of $M(R, \mathbb{C}_q^2)$ become Hopf algebras by adjoining the inverse of the q-determinant. Moreover, our constructions are quite general and can be applied similarly to quantum-braided planes associated to other R-matrices than R_{su_2}.

Finally, we have a similar situation for fermionic quantum planes $\mathbb{C}_q^{0|2}$. The natural braiding is [8]

$$\begin{aligned}&\Psi(\theta\otimes\theta)=-\theta\otimes\theta,\quad \Psi(\vartheta\otimes\vartheta)=-\vartheta\otimes\vartheta,\quad \Psi(\theta\otimes\vartheta)=-q^{-1}\vartheta\otimes\theta,\\ &\Psi(\vartheta\otimes\theta)=-q^{-1}\theta\otimes\vartheta+(q^{-2}-1)\vartheta\otimes\theta,\end{aligned}$$

which extends to

$$\begin{aligned}&\Psi(\theta\vartheta\otimes\theta)=q^{-1}\theta\otimes\theta\vartheta,\qquad \Psi(\theta\vartheta\otimes\vartheta)=q^{-1}\vartheta\otimes\theta\vartheta,\\ &\Psi(\theta\otimes\theta\vartheta)=q^{-1}\theta\vartheta\otimes\theta,\qquad \Psi(\vartheta\otimes\theta\vartheta)=q^{-1}\theta\vartheta\otimes\vartheta,\\ &\Psi(\theta\vartheta\otimes\theta\vartheta)=q^{-2}\theta\vartheta\otimes\theta\vartheta.\end{aligned}$$

In the basis used in Proposition 3.2, the corresponding R-matrix is

$$R=\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -q^{-1} & 0 & q^{-2}-1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -q^{-1} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-2} \end{pmatrix}$$

for all indices in the range 1,2,3. From this one may compute, in particular, that

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

in $M_0(\mathbb{C}_q^{0|2})$ (see below Proposition 3.2) now form a quantum matrix. The remaining generators α, β form a fermionic quantum plane, and $M_0(R, \mathbb{C}_q^{0|2})$ consists in this way of $M_q(2)$ and $\mathbb{C}_q^{0|2}$, with the zero product between their nontrivial generators.

4.2. Braided comeasuring bialgebras

Finally, we give braided versions of all our comeasuring constructions. From a categorical point of view we can fix any braided category in which our algebra lives and look for the universal comeasuring object in this braided category. The result will now be a braided group [12] or bialgebra in the braided category.

For the abstract aspect of these constructions, we use the diagrammatic notation for 'braided algebra' (due to the author) in which we write products etc. as nodes $\cdot = \cup$ and 'wire up' our algebraic operation using $\Psi = \times$, $\Psi^{-1} = \times$ as necessary. Note that the use of 'flow charts' to express algebraic or other operations is nothing new – it is use routinely by physicists as Feynman diagrams and by engineers as wiring diagrams, the new feature in braided mathematics [11,19] is to adapt the notation to algebraic constructions in braided categories, where under and over crossings are nontrivial and distinct operators. We use here the coherence theorem for braided categories [20].

In particular, using the braiding, one has a braided tensor product algebra $A\underline{\otimes}B$ for A, B algebras in the category. In a concrete setting it is $(a \otimes b)(c \otimes d) = a\Psi(b \otimes c)d$, but more generally it is defined diagrammatically, see [8]. We define a comeasuring of A as a pair (B, β) where $\beta : A \to A\underline{\otimes}B$ is a morphism and an algebra map.

Proposition 4.5. *If A is an algebra in a braided category, the universal comeasuring object $\underline{M}_1(A)$ is a braided group (a bialgebra in the braided category).*

Proof. This is shown in Fig. 1. In part (a) we write the definition of comeasuring in diagrammatic form. In part (b) we check that comeasurings are closed under tensor product. The lower box is the braided tensor product algebra $B\underline{\otimes}B$. Hence if $(\underline{M}, \beta_U)$ is the universal object, then $(\underline{M} \otimes \underline{M}, (\beta_U \otimes \mathrm{id})\beta_U)$ also comeasures, hence there is an induced algebra map $\Delta : \underline{M} \to \underline{M}\underline{\otimes}\underline{M}$. Part (c) checks that it is coassociative. □

As before, we have quotients $\underline{M}(A)$ and $\underline{M}_0(A)$ (the latter when 1 is split). We have explicit formulae in the R-matrix case. For convenience we assume that R is bi-invertible in the sense that $\tilde{R} = ((R^{t_2})^{-1})^{t_2}$ exists, where t_2 denotes transposition in the second matrix factor.

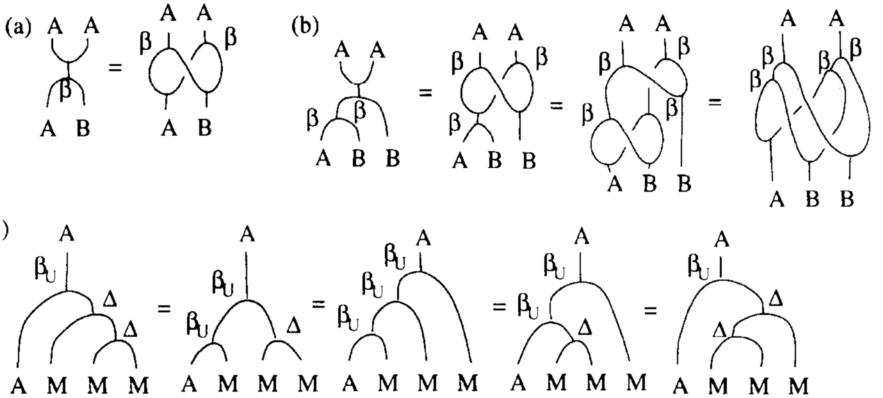

Fig. 1. (a) Comeasuring property of (B, β) (b) proof that $B\underline{\otimes}B$, $(\beta \otimes \mathrm{id})\beta$ is another comeasuring; (c) Proof that Δ is coassociative.

Proposition 4.6. *If A is a finite-dimensional braided algebra with braiding determined by a biinvertible R-matrix, then the universal comeasuring braided group has the explicit form $\underline{M}_1(R, A)$ with generators 1 and $u^i{}_j$ and the relations, coalgebra and braiding*

$$c_{ij}{}^a u^k{}_a = c_{ab}{}^k R^{-1a}{}_c{}^b{}_d u^c{}_e R^e{}_i{}^d{}_f u^f{}_j$$

$$\Delta u^i{}_j = u^i{}_a \otimes u^a{}_j, \qquad \epsilon(u^i{}_j) = \delta^i{}_j,$$

$$\Psi(u^i{}_j \otimes u^k{}_l) = u^m{}_n \otimes u^r{}_s R^i{}_a{}^d{}_m R^{-1a}{}_r{}^n{}_b R^s{}_c{}^b{}_l \tilde{R}^c{}_j{}^k{}_d.$$

The braided coaction and the braiding with A is

$$\beta_U(e_j) = e_a \otimes u^a{}_j, \qquad \Psi(u^i{}_j \otimes e_k) = e_m \otimes u^a{}_b R^{-1i}{}_a{}^m{}_n R^b{}_j{}^n{}_k,$$

$$\Psi(e_k \otimes u^i{}_j) = u^a{}_b \otimes e_m \tilde{R}^n{}_k{}^i{}_a R^m{}_n{}^b{}_j.$$

In compact form, the bialgebra structure is

$$\mathbf{u}_3 c_{12}{}^3 = c_{12}{}^3 R_{12}^{-1} \mathbf{u}_1 R_{12} \mathbf{u}_2, \quad \Delta \mathbf{u} = \mathbf{u} \otimes \mathbf{u}, \quad \epsilon \mathbf{u} = id,$$

$$\Psi(R_{12}^{-1} \mathbf{u}_1 \otimes R_{12} \mathbf{u}_2) = \mathbf{u}_2 R_{12}^{-1} \otimes \mathbf{u}_1 R_{12}.$$

Proof. That we obtain here a bialgebra follows from the preceding proposition once we have established the universal property. Before doing this we first outline, for completeness, the direct algebraic proof. For this, we have to check that the coproduct extends as an algebra map to the braided tensor product with the stated braiding Ψ. Thus,

$$\begin{aligned}
&\Delta(c_{ab}{}^k R^{-1a}{}_c{}^b{}_d u^c{}_e R^e{}_i{}^d{}_f u^f{}_j) \\
&= c_{ab}{}^k R^{-1a}{}_c{}^b{}_d R^e{}_i{}^d{}_f (u^c{}_p \otimes u^p{}_e)(u^f{}_q \otimes u^q{}_j) \\
&= c_{ab}{}^k R^{-1a}{}_c{}^b{}_d R^e{}_i{}^d{}_f u^c{}_p u^m{}_n \otimes u^r{}_s u^q{}_j R^p{}_u{}^z{}_m R^{-1u}{}_r{}^n{}_v R^s{}_w{}^v{}_q \tilde{R}^w{}_e{}^f{}_z \\
&= c_{ab}{}^k R^{-1a}{}_c{}^b{}_d u^c{}_p R^p{}_u{}^d{}_m u^m{}_n \otimes u^r{}_s u^q{}_j R^{-1u}{}_r{}^n{}_v R^s{}_i{}^v{}_q \\
&= c_{un}{}^a u^k{}_a \otimes u^r{}_s R^{-1u}{}_r{}^n{}_v u^q{}_j R^s{}_i{}^v{}_q \\
&= c_{ij}{}^a u^k{}_b \otimes u^b{}_a = \Delta c_{ij}{}^a u^k{}_a.
\end{aligned}$$

Or in the compact notation, this proof reads

$$\begin{aligned}
&\Delta(c_{12}{}^3 R_{12}^{-1} \mathbf{u}_1 R_{12} \mathbf{u}_2) = c_{12}{}^3 R_{12}^{-1} (\mathbf{u}_1 \otimes \mathbf{u}_1) R_{12} (\mathbf{u}_2 \otimes \mathbf{u}_2) \\
&= c_{12}{}^3 R_{12}^{-1} \mathbf{u}_1 \Psi(\mathbf{u}_1 R_{12} \otimes \mathbf{u}_2) \mathbf{u}_2 \\
&= c_{12}{}^3 R_{12}^{-1} \mathbf{u}_1 R_{12} \mathbf{u}_2 \otimes R_{12}^{-1} \mathbf{u}_1 R_{12} \mathbf{u}_2 \\
&= \mathbf{u}_3 c_{12}{}^3 \otimes R_{12}^{-1} \mathbf{u}_1 R_{12} \mathbf{u}_2 \\
&= \mathbf{u}_3 \otimes \mathbf{u}_3 c_{12}{}^3 = \Delta \mathbf{u}_3 c_{12}{}^3.
\end{aligned}$$

We also have to check that Ψ is itself well-defined when extended to products by functoriality (i.e. such that $\underline{M}_1(R, A)$ is a braided algebra). Here a direct proof is too complex to write out in explicit terms and we give it only with the compact notation. Thus,

$$\begin{aligned}
&\Psi(\mathbf{u}_1 R_{12} \otimes c_{34}{}^2 R_{34}{}^{-1}\mathbf{u}_3 R_{34}\mathbf{u}_4)\\
&\quad= \Psi(\mathbf{u}_1 \otimes c_{34}{}^2 R_{14} R_{13} R_{34}^{-1}\mathbf{u}_3 R_{34}\mathbf{u}_4)\\
&\quad= c_{34}{}^2 R_{34}^{-1}\Psi(\mathbf{u}_1 \otimes R_{13}R_{14}\mathbf{u}_3 R_{34}\mathbf{u}_4)\\
&\quad= c_{34}{}^2 R_{34}^{-1} R_{13}\mathbf{u}_3 R_{13}^{-1}\Psi(\mathbf{u}_1 R_{13} \otimes R_{14}R_{34}\mathbf{u}_4)\\
&\quad= c_{34}{}^2 R_{34}^{-1} R_{13}\mathbf{u}_3 R_{13}^{-1} R_{34}\Psi(\mathbf{u}_1 \otimes R_{14}\mathbf{u}_4)R_{13}\\
&\quad= c_{34}{}^2 R_{34}^{-1} R_{13}\mathbf{u}_3 R_{13}^{-1} R_{34}R_{14}\mathbf{u}_4 R_{14}^{-1} \otimes \mathbf{u}_1 R_{14}R_{13}\\
&\quad= c_{34}{}^2 R_{34}^{-1} R_{13}R_{14}\mathbf{u}_3 R_{34}R_{13}^{-1}\mathbf{u}_4 R_{14}^{-1} \otimes \mathbf{u}_1 R_{14}R_{13}\\
&\quad= c_{34}{}^2 R_{14}R_{13}R_{34}^{-1}\mathbf{u}_3 R_{34}\mathbf{u}_4 R_{13}^{-1}R_{14}^{-1} \otimes \mathbf{u}_1 R_{14}R_{13}\\
&\quad= R_{12}c_{34}{}^2 R_{34}^{-1}\mathbf{u}_3 R_{34}\mathbf{u}_4 R_{13}^{-1}R_{14}^{-1} \otimes \mathbf{u}_1 R_{14}R_{13}\\
&\quad= R_{12}\mathbf{u}_2 c_{34}{}^2 R_{13}^{-1}R_{14}^{-1} \otimes \mathbf{u}_1 R_{14}R_{13}\\
&\quad= R_{12}\mathbf{u}_2 \otimes R_{12}^{-1}c_{34}{}^2\mathbf{u}_1 R_{14}R_{13}\\
&\quad= R_{12}\mathbf{u}_2 \otimes R_{12}^{-1}\mathbf{u}_1 R_{12}c_{34}{}^2 = \Psi(\mathbf{u}_1 R_{12} \otimes \mathbf{u}_2 c_{34}{}^2)
\end{aligned}$$

using repeatedly the QYBE and one of the two covariance conditions (36). On the other side, we have

$$\begin{aligned}
&\Psi(c_{12}{}^3 R_{12}^{-1}\mathbf{u}_1 R_{12}\mathbf{u}_2 \otimes R_{14}R_{24}\mathbf{u}_4)\\
&\quad= c_{12}{}^3 R_{12}^{-1}\Psi(\mathbf{u}_1 R_{12}R_{14}\mathbf{u}_2 \otimes R_{24}\mathbf{u}_4)\\
&\quad= c_{12}{}^3 R_{12}^{-1}\Psi(\mathbf{u}_1 R_{12}R_{14}R_{24} \otimes \mathbf{u}_4)R_{24}^{-1}\mathbf{u}_2 R_{24}\\
&\quad= c_{12}{}^3 R_{12}^{-1} R_{24}\Psi(\mathbf{u}_1 R_{14} \otimes \mathbf{u}_4)R_{12}R_{24}^{-1}\mathbf{u}_2 R_{24}\\
&\quad= c_{12}{}^3 R_{12}^{-1} R_{24}R_{14}\mathbf{u}_4 R_{14}^{-1} \otimes \mathbf{u}_1 R_{14}R_{12}R_{24}^{-1}\mathbf{u}_2 R_{24}\\
&\quad= c_{12}{}^3 R_{14}R_{24}R_{12}^{-1}\mathbf{u}_4 R_{14}^{-1} \otimes R_{24}^{-1}\mathbf{u}_1 R_{12}R_{14}\mathbf{u}_2 R_{24}\\
&\quad= R_{34}\mathbf{u}_4 c_{12}{}^3 R_{12}^{-1}R_{14}^{-1}R_{24}^{-1} \otimes \mathbf{u}_1 R_{12}R_{14}\mathbf{u}_2 R_{24}\\
&\quad= R_{34}\mathbf{u}_4 c_{12}{}^3 R_{24}^{-1}R_{14}^{-1}R_{12}^{-1} \otimes \mathbf{u}_1 R_{12}R_{14}\mathbf{u}_2 R_{24}\\
&\quad= R_{34}\mathbf{u}_4 R_{34}^{-1}c_{12}{}^3 R_{12}^{-1} \otimes \mathbf{u}_1 R_{12}R_{14}\mathbf{u}_2 R_{24}\\
&\quad= R_{34}\mathbf{u}_4 R_{34}^{-1} \otimes \mathbf{u}_3 c_{12}{}^3 R_{14}R_{24}\\
&\quad= R_{34}\mathbf{u}_4 R_{34}^{-1} \otimes \mathbf{u}_3 R_{34}c_{12}{}^3\\
&\quad= \Psi(\mathbf{u}_3 R_{34} \otimes \mathbf{u}_4 c_{12}{}^3) = \Psi(\mathbf{u}_3 c_{12}{}^3 \otimes R_{14}R_{24}\mathbf{u}_4)
\end{aligned}$$

using the QYBE and the other half of (36). The proof for higher products is similar and the general case follows by induction.

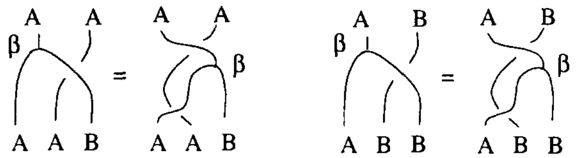

Fig. 2. Construction of braidings $B \otimes A \to A \otimes B$ and hence $B \otimes B \to B \otimes B$.

Finally, we verify the universal property. Thus, if B is a braided comeasuring on A, we define $\pi(u^i{}_j) \in B$ as $\beta(e_j) = e_a \otimes \pi(u^a{}_j)$. That this is a morphism implies that the braiding with A may be computed before or after applying β. This is shown in Fig. 2. Thus

$$e_a \otimes \Psi(\pi(u^a{}_i) \otimes e_j) = \Psi^{-1}(e_b \otimes e_c) \otimes u^c{}_a R^a{}_i{}^b{}_j$$
$$= e_m \otimes e_n R^{-1m}{}_c{}^n{}_b \otimes u^c{}_a R^a{}_i{}^b{}_j$$

which tells us that the braiding $\Psi : B \otimes A \to A \otimes B$ is compatible via π with the braiding of $\underline{M}(R, A)$ with A. The braiding with A on the other side similarly gives compatibility of $\Psi : A \otimes B \to B \otimes A$. Once these are known then the braiding of A with $\pi(u^j{}_k) \in B$ before and after β yields compatibility with $\Psi : B \otimes B \to B \otimes B$. This is shown on the right in Fig. 2. In this way, the braiding on $\pi(u^i{}_j)$ has the same form as the braiding on $u^i{}_j$. Next, the assumption that β is an algebra map to $A\underline{\otimes}B$ translates as

$$\beta(e_i e_j) = c_{ij}{}^a e_k \otimes \pi(u^k{}_a)$$
$$= (e_a \otimes \pi(u^a{}_i))(e_b \otimes \pi(u^b{}_j)) = e_a e_c \otimes \pi(u^d{}_e) R^{-1a}{}_d{}^c{}_f R^e{}_i{}^f{}_d \pi(u^b{}_j).$$

Thus $u^i{}_j \mapsto \pi(u^i{}_j)$ extends as an algebra map $\pi : \underline{M}(R, A) \to B$ and a morphism in the braided category. □

Finally, we recall that the quantum matrices $A(R)$ above have a braided group version $B(R)$ [12] defined by a matrix of generators $\mathbf{u} = (u^i{}_j)$ and relations $R_{21}\mathbf{u}_1 R\mathbf{u}_2 = \mathbf{u}_2 R_{21}\mathbf{u}_1 R$, forming a braided group with the above matrix coalgebra and braiding. The algebra relations here are known in other contexts too [14] and sometimes called 'reflection equations', although they have been first introduced and studied as quadratic algebras by the author under the heading of the braided matrix relations. Key properties such as its covariance properties (as a braided algebra), the braided coproduct, results about the Poincaré series, etc. were introduced in [12].

On the other hand, the quantum and braided matrices are closely tied by a theory of transmutation which relates their products, e.g.

$$\mathbf{t} = \mathbf{u}, \qquad R_{12}\mathbf{t}_1\mathbf{t}_2 = \mathbf{u}_1 R\mathbf{u}_2. \tag{43}$$

See [8,19] for an introduction to this transmutation theory of braided groups.

Corollary 4.7. *The transmutation of $M_1(R, A)$ is the braided comeasuring bialgebra $\underline{M}_1(R, A)$ generated by 1 and $u^i{}_j$ with the braided matrix relations $R_{21}\mathbf{u}_1 R\mathbf{u}_2 = \mathbf{u}_2 R_{21}\mathbf{u}_1 R$*

and the further relations $\mathbf{u}_3 c_{12}{}^3 = c_{12}{}^3 R_{12}^{-1} \mathbf{u}_1 R_{12} \mathbf{u}_2$ *and coalgebra of* $\underline{M}_1(A)$ *from the preceding proposition.*

Proof. This is immediate from (43) applied to the relations $\mathbf{t}_3 c_{12}{}^3 = c_{12}{}^3 \mathbf{t}_1 \mathbf{t}_2$ of $M(R, A)$. □

One may then proceed to apply the extensive theory of braided groups to $\underline{M}_1(A)$, $\underline{M}_1(R, A)$ and their unital quotients. For example, associated to any braided group in the category of comodules of a quantum group H (which is essentially the situation above, with H obtained from $A(R)$), one has ordinary bialgebras $\underline{M}_1(A) \rtimes H$, etc., by the bosonisation construction.

As a simple example, we consider $\mathbb{C}[x]$ as a braided algebra with R-matrix (40). In this case the relations from Proposition 4.6 are

$$u^k{}_{i+j} = \sum_{a+b=k} u^a{}_i u^b{}_j q^{(i-a)b}.$$

Hence $\underline{M}(\mathbb{C}[x]) = \mathbb{C}\langle u_i | i \in \mathbb{Z}_+\rangle$ is the free algebra as in Section 3.1, but the other generators (and hence the matrix coalgebra) are given from these by

$$u^i{}_j = \sum_{a_1+\cdots+a_j=i} u_{a_1} \cdots u_{a_j} q^{-(i-a_1)-\sum_{s=2}^{j}(a_1+\cdots a_{s-1}{}^{-s})a_s},$$

which is a q-deformation of (5) as a braided group. The braiding is

$$\Psi(u_i \otimes u_j) = u_j \otimes u_i q^{(i-1)(j-1)}.$$

Similarly, the braided-commutativity in Corollary 4.7 in this simplest example reduces to the usual commutativity relations. For example, $\underline{M}_0(R, \mathbb{C}[x]) = \mathbb{C}[u_i | i \in \mathbb{N}]$ is the same algebra as $\mathrm{Diff}_0(\mathbb{C}[x])$ in the classical case, but the coalgebra is q-deformed and provides a nontrivial braided group structure on this algebra.

References

[1] M.E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[2] M. Batchelor, Difference functions, measuring coalgebras, and quantum group-like objects, Preprint, 1990.
[3] S. Wang, Quantum symmetry groups of finite spaces, Berkeley preprint, 1997.
[4] H. Albuquerque, S. Majid, Quasialgebra structure of the octonions, Preprint, 1997.
[5] T. Brzeziński, S. Majid, Quantum group gauge theory on quantum spaces. Commun. Math. Phys. 157 (1993) 591–638; Erratum 167 (1995) 235.
[6] S. Majid, Quantum and braided group riemannian geometry, Preprint, 1997.
[7] Yu.I. Manin, Quantum groups and non-commutative geometry. Technical report, Centre de Recherches Math, Montreal, 1988.
[8] S. Majid, Foundations of Quantum Group Theory, Cambridge University Press, Cambridge, 1995.
[9] T. Breziński, S. Majid, Quantum differentials and the q-monopole revisited, Acta Appl. Math. 1998, to appear.
[10] S. Majid, Braided momentum in the q-Poincaré group, J. Math. Phys. 34 (1993) 2045–2058.
[11] S. Majid, Braided groups and algebraic quantum field theories, Lett. Math. Phys. 22 (1991) 167–176.

[12] S. Majid, Examples of braided groups and braided matrices, J. Math. Phys. 32 (1991) 3246–3253.
[13] V.G. Drinfeld, QuasiHopf algebras, Leningrad Math. J. 1 (1990) 1419–1457.
[14] L.D. Faddeev, N.Yu. Reshetikhin, L.A. Takhtajan, Quantization of Lie groups and Lie algebras, Leningrad Math. J. 1 (1990) 193–225.
[15] S. Majid, Quasitriangular Hopf algebras and Yang–Baxter equations, Internat. J. Modern Physics A 5(1) (1990) 1–91.
[16] S. Majid, Solutions of the Yang-Baxter equations from braided-Lie algebras and braided groups, J. Knot Th. Ram., 4 (1995) 673–697.
[17] S. Majid, Braided geometry of the conformal algebra, J. Math. Phys. 37 (1996) 6495–6509.
[18] S. Majid, M. Markl, Glueing operation for R-matrices, quantum groups and link invariants of Hecke type, Math. Proc. Camb. Phil. Soc. 119 (1996) 139–166.
[19] S. Majid, Beyond supersymmetry and quantum symmetry (an introduction to braided groups and braided matrices), in: M-L. Ge, H.J. de Vega, (Eds.), Quantum Groups, Integrable Statistical Models and Knot Theory, World Scientific 1993, pp. 231–282.
[20] A. Joyal, R. Street, Braided monoidal categories, Mathematics Reports 86008, Mac-quarie University, 1986.